Journal of Positive School Psychology http://journalppw.com
2022, Vol. 6, No. 5, 3274-3288

Modeling Lifetime Data With Weibull-Exponentiated Exponential Distribution (W-Ee)

Ayat Khaled Sagheer[1], Kareema Abed AL-Kadim[2]

[1,2] *Mathematics Department, College of Education for pure Sciences, University of Babylon/Iraq*
ayat.sqhuer@student.uobabylon.equ.iq , Kareema.kadim@yahoo.com

Abstract

In this study we obtained a new four- parameters continuous distribution called the Weibull-Exponentiated Exponential distribution $(\mathrm{W-EE})$ for modeling lifetime data that created in a new way by using two life distributions , the results show that the proposed generalization performs better than the other known extensions of the Weibull distribution considered for the study.

Keywords : Weibull-Exponentiated Exponential distribution , Reliability analysis , Moments , Order statistics , Reni entropy , Parameters Estimation , applications.

Introduction :

In a statistics analysis and reliability, the family of distributions such as the exponential distribution, Weibull distribution, normal distribution, and gamma distribution has proven to be of great importance in the data modeling of life. In reliability , the family of Weibull distributions of great importance in different models proved monotonous failure rates. Statistically , This family contains distributions that can represent data characterized by increasing, decreasing, and exponential failure rates. In real practice, many unexpected data also appear that weaken the proper representation using the failure models of this and other families in the field of reliability . In (1993) Mudholkar and Srivastava introduced The Exponentiated Weibull family [1]. Mudholkar et al. They provided applications of the Exponentiated Weibull distribution(EW) in reliability In (1995).In our study of this research, we presented a new distribution in a new way by using two life distributions, namely the Weibull distribution and the exponential distribution . Our goal in generalization is to create flexible distributions that fit most life data.

Modeling Lifetime data with Weibull-Exponentiated Exponential distribution $(\mathbf{W-EE})$.

We studied the 4-parameters Weibull - Exponentiated Exponential distribution $\mathrm{W-EE}\ (a, b, \alpha, \lambda, a, b)$ distribution .

The CDF and PDF of $(\mathbf{W-EE})$ Distribution.

We can find the Cdf and Pdf of $(\mathrm{W-EE})$ distribution by using the Cdf of the Weibull distribution that given by :

$$F(X; a, b) = \left[1 - e^{-ax^{b}}\right] , \ x > 0 , a > 0 , b > 0 \qquad (1)$$

And the cdf of the (EE) distribution that given by :

$F(X;\alpha,\lambda) = [1 - e^{-\alpha x}\ ]^{\lambda}\ ,\ x > 0\ ,\alpha > 0\ ,\lambda > 0$ (2)

as the follows :

We multiply (x^b) in equation (1) by equation (2) lets get cdf of $(W - EE)$ distribution as the following equation :

$$F(X;a,b,\alpha,\lambda) = 1 - e^{-ax^b[1-e^{-\alpha x}\ ]^{\lambda}}, x > 0, \alpha > 0, \lambda > 0, a > 0, b > 0 \qquad (3)$$

And the pdf congruous to equation (3) is :

$$f(X;a,b,\alpha,\lambda) = ae^{-ax^b[1-e^{-\alpha x}]^{\lambda}}\left[x^b\alpha\lambda e^{-\alpha x}[1 - e^{-\alpha x}]^{\lambda-1} + bx^{b-1}[1 - e^{-\alpha x}]^{\lambda}\right]\ ,$$

$$x > 0\ ,\qquad \alpha,\lambda,a,b > 0$$

(4)

$$f(X;a,b,\alpha,\lambda) = ae^{-ax^b[1-e^{-\alpha x}]^{\lambda}}\left[\left[x^b[1 - e^{-\alpha x}]^{\lambda}\right][\alpha\lambda[1 - e^{-\alpha x}]^{-1}e^{-\alpha x} + bx^{-1}]\right]$$

$$x > 0\ ,\qquad \alpha,\lambda,a,b > 0$$

(5)

$$f(X;a,b,\alpha,\lambda) = ax^b e^{-ax^b[1-e^{-\alpha x}]^{\lambda}}[1 - e^{-\alpha x}]^{\lambda}\left[\frac{\alpha\lambda}{e^{\alpha x} - 1} + \frac{b}{x}\right]$$

$$x > 0\ ,\qquad \alpha,\lambda,a,b > 0$$

(6)

Where b and λ are shape parameters and a and α are scale parameters.

Now we prove it is pdf

1. $f(X;a,b,\alpha,\lambda) \geq 0$
2. $\int_0^{\infty} f(X;a,b,\alpha,\lambda)dx = 1$

That is

1. $f(X;a,b,\alpha,\lambda) \geq 0$

In (EE) distribution $x \in [0,\infty)$ then in W-EE distribution $x \in [0,\infty)$

When $x \to 0$, then $f(X;\alpha,\lambda,a,b) = 0$ and as $x \to \infty$, then $f(x;\alpha,\lambda,a,b) = \infty$

$$\therefore\ f(x;\alpha,\lambda,a,b) \geq 0$$

2. $\int_0^{\infty} f(X;a,b,\alpha,\lambda) =$

$$= \int_0^{\infty} ae^{-ax^b[1-e^{-\alpha x}]^{\lambda}}\left[x^b\alpha\lambda[1 - e^{-\alpha x}]^{\lambda-1}e^{-\alpha x} + bx^{b-1}[1 - e^{-\alpha x}]^{\lambda}\right]$$

(7)

$$= \left[e^{-ax^b[1-e^{-\alpha x}]^{\lambda}}\right]\Big|_0^{\infty} = 1$$

(8)

The plots of the cumulative distribution function $f(X;a,b,\alpha,\lambda)$ are given by the following figures

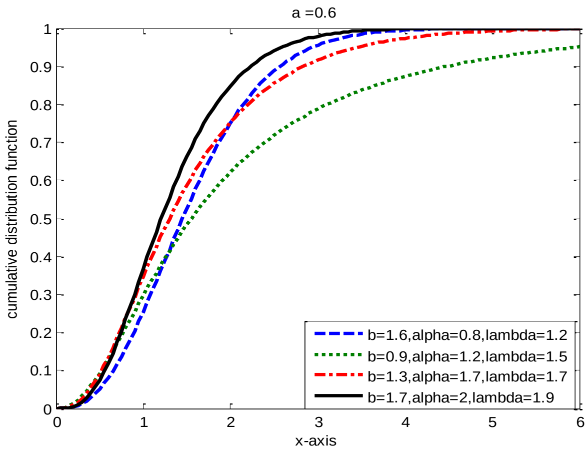

Figure (1) : The Cdf of $W-EE$ distribution with the parameters $a = 0.6$;

$b = (1.6,0.9,1.3,1.7)$; $\alpha = (0.8 , 1.2 , 1.7, 2)$; $\lambda = (1.2 , 1.5 , 1.7 , 1.9)$.

Figures (1) indicate that the cdf of the $W-EE$ is non-decreasing with increasing x and the parameters α , λ , a , b .

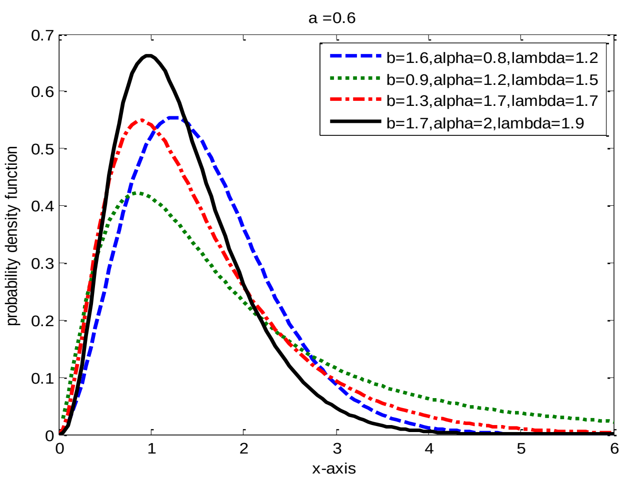

Figure (2) : The Pdf of $W-EE$ distribution with the parameters $a = 0.6$;

$b = (1.6,0.9,1.3,1.7)$; $\alpha = (0.8 , 1.2 , 1.7, 2)$; $\lambda = (1.2 , 1.5 , 1.7 , 1.9)$.

Figures (2), indicate that the $W-EE$ family generates various shapes such as symmetrical , right skewed and reversed-J .

The Limit of The Pdf and Cdf of W-EE .

The limit of the pdf is given as follows :

$$\lim_{x\to 0} f(X;\alpha,\lambda,a,b) = a\lim_{x\to 0}\left[e^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[\left[x^b\alpha\lambda[1-e^{-\alpha x}]^{\lambda-1}\right]e^{-\alpha x}+bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]\right]=0 \quad (9)$$

$$\lim_{x\to \infty} f(X;\alpha,\lambda,a,b) = a\lim_{x\to \infty}\left[e^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[\left[x^b\alpha\lambda[1-e^{-\alpha x}]^{\lambda-1}\right]e^{-\alpha x}+bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]\right]=0 \quad (10)$$

This is the pdf approaches to 0 as x approach to 0 or ∞ and that is shown Figure (4) .

The limit of cdf is given as follows :

$$\lim_{x\to 0} F(X;\alpha,\lambda,a,b) = \lim_{x\to 0}\left[1-e^{-ax^b[1-e^{-\alpha x}]^\lambda}\right] = 0 \quad (11)$$

$$\lim_{x\to \infty} F(X;\alpha,\lambda,a,b) = \lim_{x\to \infty}\left[1-e^{-ax^b[1-e^{-\alpha x}]^\lambda}\right] = 1 \quad (12)$$

This is the cdf approaches to 0 as x approach to 0 and ∞ and the cdf approaches to 1 as x approach to ∞ and that is shown figure (1) .

Reliability Analysis

In this section , we introduce the reliability (survival) function $\bar{F}(X)$, Hazard function $r(X)$ and cumulative hazard rate function $r(X)$ and cumulative hazard rate function $H(X)$ of $\mathbf{X \sim W-EE(\alpha,\lambda,a,b)}$.

Reliability Function

The reliability (survivor) function of $\mathbf{X \sim W-EE(\alpha,\lambda,a,b)}$ is defined as follows .

$$\bar{F}(X;\alpha,\lambda,a,b) = 1 - F(X;\alpha,\lambda,a,b)$$

$$= 1-\left[1-e^{-ax^b[1-e^{-\alpha x}]^\lambda}\right] = \left[e^{-ax^b[1-e^{-\alpha x}]^\lambda}\right] \quad (13)$$

The plots of the reliability function $\bar{F}(x;\alpha,\lambda,a,b)$ are given by the following figures .

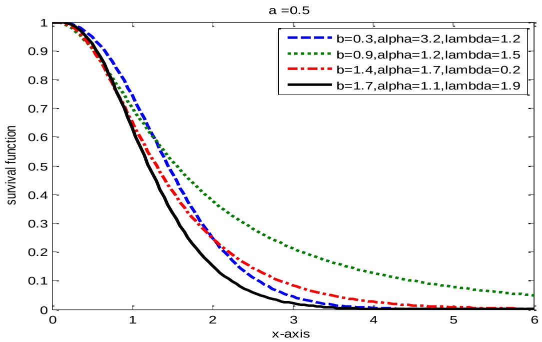

Figure (3) : The $\bar{F}(X)$ of $W-EE$ distribution with the parameters $a=0.5$;

$b=(0.3,0.9,1.4,1.7)$; $\alpha=(3.2\,,1.2\,,1.7,1.1)$; $\lambda=(1.2\,,1.5\,,0.2\,,1.9)$.

Figures (3), indicate that the $S(X)$ of $W-EE$ is decrease function .

Hazard Function:

The hazard function describes the likelihood that a system or an individual will not fail after a given time .

For a continuous distribution with pdf $f(X)$, the hazard function , of $X\sim W-EE(\alpha,\lambda,a,b)$ is known as follows :

$$h(X;\alpha,\lambda,a,b)=\frac{f(X;\alpha,\lambda,a,b)}{\bar{F}(X;\alpha,\lambda,a,b)}$$

$$=\frac{ae^{-ax^{b}[1-e^{-\alpha x}]^{\lambda}}\left[x^{b}\alpha\lambda[1-e^{-\alpha x}]^{\lambda-1}e^{-\alpha x}+bx^{b-1}[1-e^{-\alpha x}]^{\lambda}\right]}{\left[e^{-ax^{b}[1-e^{-\alpha x}]^{\lambda}}\right]} \qquad (14)$$

The plots of the hazard function $\bar{F}(X;\alpha,\lambda,a,b)$ are given by the following figures .

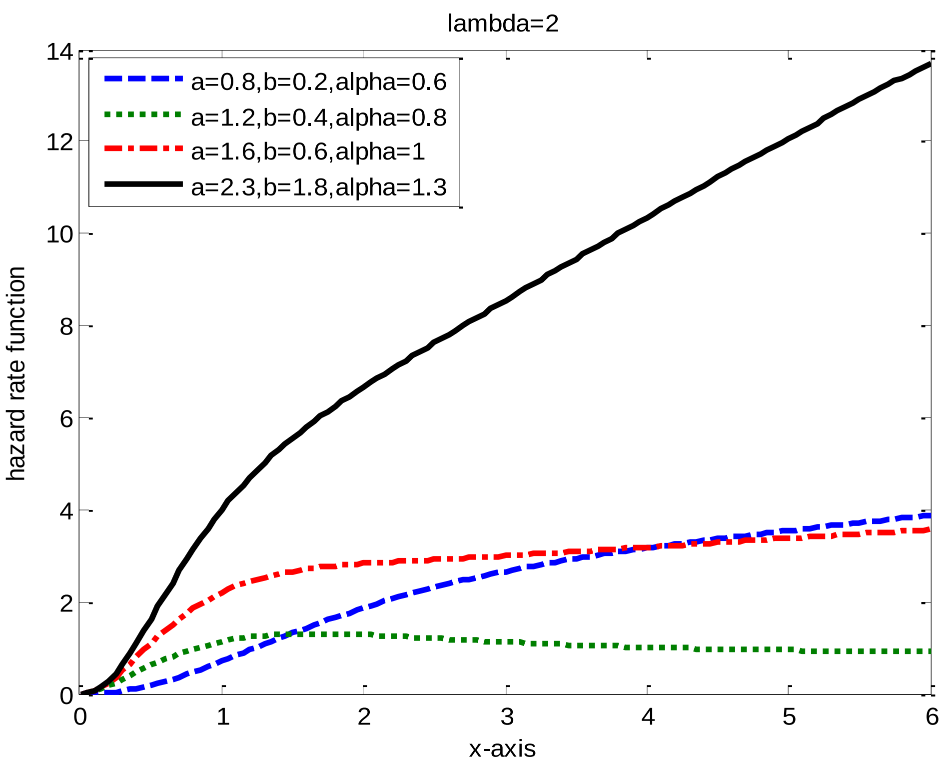

Figure (4) : The hazard function of $W-EE$ distribution with the parameters

$a=(0.8,1.2,1.6,2.3)$; $b=(0.2,0.4,0.6,1.8)$; $\alpha=(0.6,0.8,1,1.3)$; $\lambda=(2)$.

The Reverse Hazard Function

The reverse hazard function of $\mathbf{X \sim EE-W(\alpha,\lambda,a,b)}$ is defined as follows

$$r(x,\alpha,\lambda,a,b)=\frac{f(x;\alpha,\lambda,a,b)}{F(x;\alpha,\lambda,a,b)}$$

$$=\frac{ae^{-ax^{b}[1-e^{-\alpha x}]^{\lambda}}\left[x^{b}\alpha\lambda[1-e^{-\alpha x}]^{\lambda-1}e^{-\alpha x}+bx^{b-1}[1-e^{-\alpha x}]^{\lambda}\right]}{\left[1-e^{-ax^{b}[1-e^{-\alpha x}]^{\lambda}}\right]} \qquad (15)$$

The plots of the reverse hazard function of $W-EE$ are given by the following figures

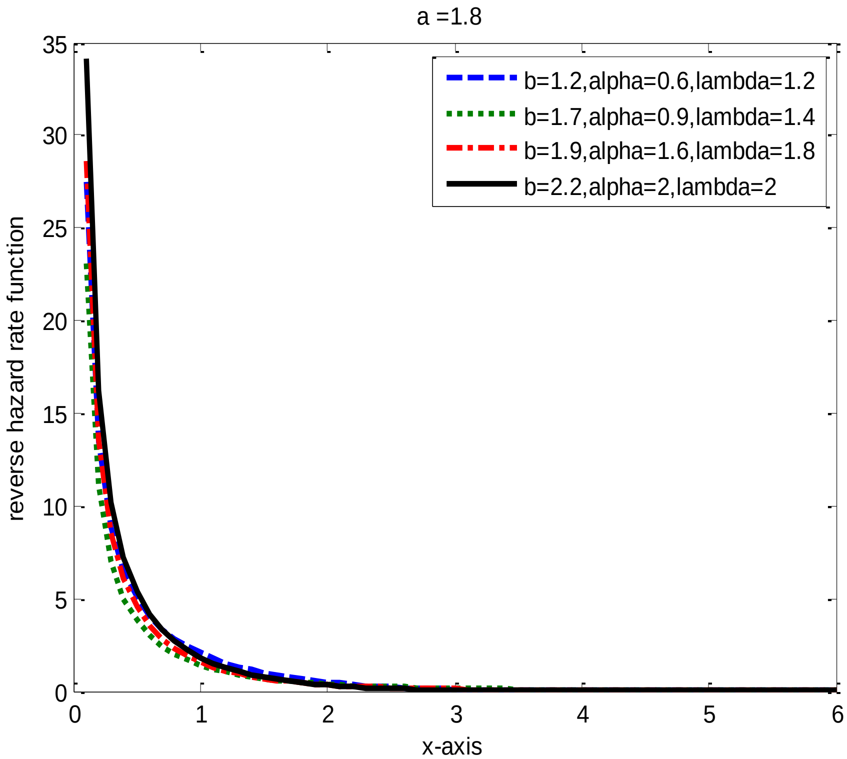

Figure (13) : Reverse ha. fun. of $W-EE$ distribution with the parameters

$a=1.8$; $b=(1.2,1.7,1.9,2.2)$; $\alpha=(0.6,0.9,1.6,2)$; $\lambda=(1.8,1.4,1.8,2)$.

The Cumulative Ha. Fun.

The cumulative hazard function of $\mathbf{X \sim W-EE(\alpha,\lambda,a,b)}$ is defined as follows

$$H(x;\alpha,\lambda,a,b)=-\ln[1-F(x,\alpha,\lambda,a,b)]$$

$$=-\ln\left[1-\left[1-e^{-ax^{b}[1-e^{-\alpha x}]^{\lambda}}\right]\right]$$

$$=-\ln\left[e^{-ax^{b}[1-e^{-\alpha x}]^{\lambda}}\right]$$

$$= ax^b[1-e^{-\alpha x}]^\lambda \qquad (16)$$

The plots of the cumulative ha. fun. of $W-EE$ are given by the following figures.

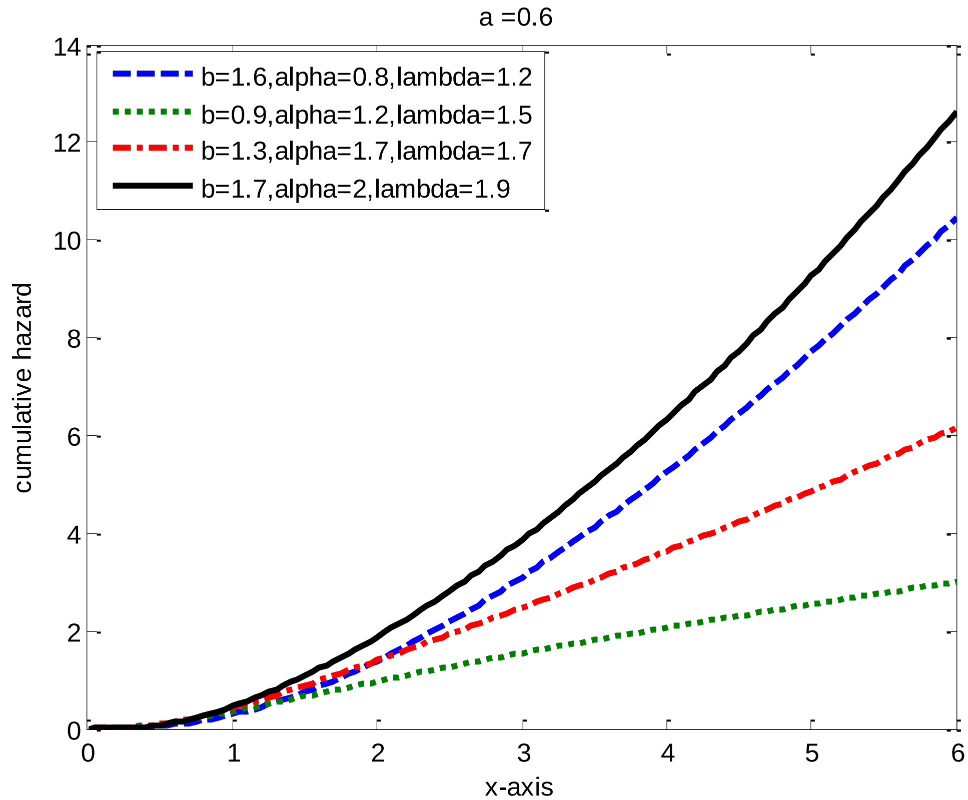

Figure (17) : The cumulative hazard of W-EE distribution with the parameters

$a=0.6$; $b=(1.6,0.9,1.3,1.7)$; $\alpha=(0.8\,,1.2,1.7,2)$; $\lambda=(1.2\,,1.5\,,1.7\,,1.9)$.

The Moments and Coefficients of Skewness , Kurtosis and Variation

The moments

We introduce the r^{th} moment about the origin , r^{th} moment about the mean and coefficient of skewness , kurtosis and variation for the $W-EE$ distribution .

$$\acute{\mu}_r = E(X^r) = \int_0^\infty x^r f(x)\,dx$$

$$= a\int_0^\infty x^r e^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[x^b\alpha\lambda[1-e^{-\alpha x}]^{\lambda-1}e^{-\alpha x} + bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]dx \qquad (17)$$

By using series expansion of $e^{-ax^b[1-e^{-\alpha x}]^\lambda}$ we get

$$e^{-ax^b[1-e^{-\alpha x}]^\lambda} = \sum_{j=0}^\infty \frac{(-1)^j a^j x^{bj}[1-e^{-\alpha x}]^j}{j!} \qquad (18)$$

We substitute (18) in (17) we get

$$E(X^r) = a\int_0^{\infty} x^r \sum_{j=0}^{\infty} \frac{(-1)^j a^j x^{bj}[1-e^{-\alpha x}]^{\lambda j}}{j!}\left[x^b \alpha\lambda[1-e^{-\alpha x}]^{(\lambda-1)} e^{-\alpha x} + b x^{(b-1)}[1-e^{-\alpha x}]^{\lambda}\right] dx \quad (19)$$

$$= a\sum_{j=0}^{\infty} \frac{(-1)^j a^j}{j!}\left[\alpha\lambda\int_0^{\infty} x^{r+b(j+1)}[1-e^{-\alpha x}]^{\lambda(j+1)-1} e^{-\alpha x} dx + b\int_0^{\infty} x^{r+b(j+1)-1}[1-e^{-\alpha x}]^{\lambda(j+1)} dx\right]$$
(20)

using binomial series expansion of $[1-e^{-\alpha x}]^{\lambda(j+1)-1}$ and $[1-e^{-\alpha x}]^{\lambda(j+1)}$

$$[1-e^{-\alpha x}]^{\lambda(j+1)-1} = \sum_{n=0}^{\infty}(-1)^n \binom{\lambda(j+1)-1}{n} e^{-\alpha n x} \quad (21)$$

$$[1-e^{-\alpha x}]^{\lambda(j+1)} = \sum_{m=0}^{\infty}(-1)^m \binom{\lambda(j+1)}{m} e^{-\alpha m x} \quad (22)$$

We substitute (20) and (21) in (19) we get

$E(X^r) =$

$$a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(r+b(j+1)+1)}{(\alpha n+\alpha)^{r+b(j+1)+1}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(r+b(j+1))}{(\alpha m)^{r+b(j+1)}} \quad (23)$$

The equation (22) is the $\mathbf{r^{th}}$ moment of the $W-EE$ distribution .

When $r = 1$ then $E(X^1) = \mu$

$$E(X^1) = \mu = a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(1+b(j+1))}{(\alpha m)^{1+b(j+1)}}$$

(24)

The equation (24) is the Expectation $\mathbf{E(X)}$ of the $\mathbf{W-EE}$ distribution .

When $r = 2$ then

$$E(X^2) = a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(3+b(j+1))}{(\alpha n+\alpha)^{3+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(2+b(j+1))}{(\alpha m)^{2+b(j+1)}}$$

(25)

$var(X) = E(X^2) - E(X)^2$

$$= \left[a\alpha\lambda \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{j+n}a^j}{j!} \binom{\lambda(j+1)-1}{n} \frac{\Gamma(3+b(j+1))}{(\alpha n+\alpha)^{3+b(j+1)}} + ab \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{j+m}a^j}{j!} \binom{\lambda(j+1)}{m} \frac{\Gamma(2+b(j+1))}{(\alpha m)^{2+b(j+1)}} \right] - \left[a\alpha\lambda \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{j+n}a^j}{j!} \binom{\lambda(j+1)-1}{n} \frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}} + ab \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{j+m}a^j}{j!} \binom{\lambda(j+1)}{m} \frac{\Gamma(1+b(j+1))}{(\alpha m)^{1+b(j+1)}} \right]^2$$

(26)

The equation (26) is the variance $\mathbf{var}(\mathbf{X})$ of the $\mathbf{W-EE}$ distribution .

$$E(X-\mu)^r = a\alpha\lambda \sum_{k=1}^{r}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+j+n}a^j\mu^k}{j!} C_k^r \binom{\lambda(j+1)-1}{n} \frac{\Gamma(r-k+bj+1)}{(\alpha n+\alpha)^{(r-k+bj+1)}} + ab \sum_{k=1}^{r}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+j+m}a^j\mu^k}{j!} C_k^r \binom{\lambda(j+1)}{m} \frac{\Gamma(r-k+b(j+1))}{(\alpha m)^{(r-k+b(j+1))}} \quad (27)$$

The equation (27) is the r^{th} center moment about the mean of the $\mathbf{W-EE}$ distribution .

Coefficient of Skewness

It is denoted by **CS** , and we can know if the distribution under study is symmetric or not , it is expressed by $CS = \frac{E(X-\mu)^3}{\left(\sqrt{var(x)}\right)^3}$ when $E(X-\mu)^3$ is center moment about the mean when $r = 3$, and $\sqrt{var(x)}$ is the square root of the variance of distribution .

$$CS = \frac{E(X-m)^3}{\left(\sqrt{var(X)}\right)^3} \quad (28)$$

$$\text{Let } CS = \frac{A}{B} \quad (29)$$

By equation (27) , put $r = 3$ then .

$$A = E(X-m)^3$$

$$= \left(a\alpha\lambda \sum_{k=1}^{3}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+j+n}a^j\mu^k}{j!} C_k^3 \binom{\lambda(j+1)-1}{n} \frac{\Gamma(4-k+bj)}{(\alpha n+\alpha)^{(4-k+bj)}} + ab \sum_{k=1}^{3}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+j+m}a^j\mu^k}{j!} C_k^3 \binom{\lambda(j+1)}{m} \frac{\Gamma(3-k+b(j+1))}{(\alpha m)^{(3-k+b(j+1))}} \right)$$

By equation (26)

$$B=\left(\sqrt{var(X)}\right)^{3}=\left(var(X)\right)^{\frac{3}{2}}$$
$$=\left(\left[a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^{j}}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(3+b(j+1))}{(\alpha n+\alpha)^{3+b(j+1)}}+ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^{j}}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}}\right]-\left[a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^{j}}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}}+ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^{j}}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(1+b(j+1))}{(\alpha n+\alpha)^{1+b(j+1)}}\right]^{2}\right)^{\frac{3}{2}}$$

Coefficient of Kurtosis

It is denoted by CK the coefficient of kurtosis measures the flatness of the top , and it is expressed by $CK=\frac{E(X-\mu)^4}{(var(X))^2}$, when $E(X-\mu)^4$ the center moment about the mean when $r=4$, var(X) the variance of distribution .

$$CK=\frac{E(X-m)^4}{(var(X))^2} \tag{30}$$

Let $CK=\frac{C}{D}$ (31)

By equation (27) , put $r=4$ then .

$$C=E(X-m)^4=\left(a\alpha\lambda\sum_{k=1}^{4}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+j+n}a^{j}\mu^{k}}{j!}C_k^4\binom{\lambda(j+1)-1}{n}\frac{\Gamma(5-k+bj)}{(\alpha n+\alpha)^{(5-k+bj)}}+ab\sum_{k=1}^{4}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+j+m}a^{j}\mu^{k}}{j!}C_k^4\binom{\lambda(j+1)}{m}\frac{\Gamma(4-k+b(j+1))}{(\alpha m)^{(4-k+b(j+1))}}\right)$$

By equation (26)

$$D=(var(X))^2$$

$$= \left(\left(a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(3+b(j+1))}{(\alpha n+\alpha)^{3+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}}\right) - \left(a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(1+b(j+1))}{(\alpha n+\alpha)^{1+b(j+1)}}\right)^2\right)$$

Coefficient of Variation

It is denoted by CV and it is defined by $CV = \frac{\sqrt{var(X)}}{E(X)}$ where $\sqrt{var(X)}$ is the square root of the variance and $E(X)$ is the expection of distribution .

$$CV = \frac{\sqrt{var(X)}}{E(X)} \tag{2.32}$$

Let $CV = \frac{E}{F}$ (2.33)

By equation (26)

$$E = \sqrt{var(X)} = (var(X))^{\frac{1}{2}}$$

$$E = \left[\left(a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(3+b(j+1))}{(\alpha n+\alpha)^{3+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}}\right) - \left(a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(1+b(j+1))}{(\alpha n+\alpha)^{1+b(j+1)}}\right)^2\right]^{\frac{1}{2}}$$

By equation (24)

$F = E(X)$

$$F = a\alpha\lambda \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(1+b(j+1))}{(\alpha n+\alpha)^{1+b(j+1)}}$$

Order statistics

In this section , the pdf of the j^{th} order statistic and the pdf of the smallest and largest order statistics of $W-EE$ distribution are derived .

Let $x_1, x_2, \dots, x_n$ a radom sample from an $W-EE$ distribution and $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ denote the order statistics obtained from this sample , then the pdf of $x_{j:n}$ is given by:

$$f_{j:n}(X,\alpha,\lambda,a,b) = \frac{n!}{(j-1)!\,(n-j)!} f(X,\alpha,\lambda,a,b)[F(X,\alpha,\lambda,a,b)]^{j-1}[1-F(X,\alpha,\lambda,a,b)]^{n-j}$$

(34)

When $f(X,\alpha,\lambda,a,b)$ is pdf of $W-EE$ distribution given by equation (4) , $F(X,\alpha,\lambda,a,b)$ is cdf of $W-EE$ distribution given by equation (3).

$$= \frac{n!}{(j-1)!\,(n-j)!}\left[ae^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[x^b\alpha\lambda e^{-\alpha x}[1-e^{-\alpha x}]^{\lambda-1} + bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]\right]\left[1 - e^{-ax^b[1-e^{-\alpha x}\ ]^\lambda}\right]^{j-1}\left[1-\left(1-e^{-ax^b[1-e^{-\alpha x}\ ]^\lambda}\right)\right]^{n-j}$$

(35)

Then the pdf of max. , min. , and the med. are explained as:

1. When $j = 1$, the pdf of minimum.

$$f_{1:n}(X,\alpha,\lambda,a,b) = \frac{n!}{(n-1)!}\left[ae^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[x^b\alpha\lambda e^{-\alpha x}[1-e^{-\alpha x}]^{\lambda-1} + bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]\right]\left[e^{-ax^b[1-e^{-\alpha x}\ ]^\lambda}\right]^{n-1}$$

(2.)

2. When $j = n$, then pdf of maximum .

$$f_{n:n}(X,\alpha,\lambda,a,b) = \frac{n!}{(n-1)!}\left[ae^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[x^b\alpha\lambda e^{-\alpha x}[1-e^{-\alpha x}]^{\lambda-1} + bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]\right]\left[1 - e^{-ax^b[1-e^{-\alpha x}\ ]^\lambda}\right]^{n-1}$$

(36)

3. When $j = m + 1$, then pdf of median .

$$f_{(m+1):n}(X,\alpha,\lambda,a,b) = \frac{n!}{m!(n-m-1)!}\left[ae^{-ax^b[1-e^{-\alpha x}]^\lambda}\left[x^b\alpha\lambda e^{-\alpha x}[1-e^{-\alpha x}]^{\lambda-1} + bx^{b-1}[1-e^{-\alpha x}]^\lambda\right]\right]\left[1 - e^{-ax^b[1-e^{-\alpha x}\ ]^\lambda}\right]^{m}\left[e^{-ax^b[1-e^{-\alpha x}\ ]^\lambda}\right]^{n-m-1} \quad (37)$$

Reni entropy

Renyi entropy is necessary for quantitative information as can be used as a measure of entanglement ,and it is also importantant in statistics and ecology as an indicator diversity.

The renyi entropy of the random variable X with pdf is known as follows : [11]

$$I_r(\delta) = \frac{1}{1-\delta}\log(\int_0^{\infty} f^{\delta}(\alpha)\, dx\,, \text{wher } \delta > 0, \delta \neq 1$$

$$= \frac{1}{1-\delta}\log(\int_0^{\infty} \left[ax^r e^{-ax^b[1-e^{-\alpha x}]^{\lambda}}\left[x^b\alpha\lambda[1-e^{-\alpha x}]^{\lambda-1}e^{-\alpha x} + bx^{b-1}[1-e^{-\alpha x}]^{\lambda}\right]\right]^{\delta} dx)\,, \text{wher } \delta > 0, \delta \neq 1 \quad (38)$$

From equation (2.27) we get the renyi entropy of x given eq. (2.) by applying the same steps for finding $\acute{\mu}_r$.

Parameters Estimation of W-EE

In this section , the two considered estimation methods (the maximum likelihood estimation and the moment method) are illustrated to estimate the for parameters of $W-EE$ distribution .

Max "Likelihood" Estimation

The parameters estimation method is the most used in the literature , we will discuss the Max likelihood "ML" estimation for the parameters of the $(W-EE)$ distribution for complete samples for a random samples ,

If $x_1, x_2, x_3, \dots \dots, x_n$ denoted random sample from the $\mathbf{X \sim W - EE}$ distribution , then the L.K.F is given by

$$L = \prod_{i=1}^{n} f(X_i, \alpha, \lambda, a, b) \quad (39)$$

Form substitution the eq.(6) into eq.(39)

$$L = \prod_{i=1}^{n}\left[ax^b e^{-ax^b[1-e^{-\alpha x}]^{\lambda}}[1-e^{-\alpha x}]^{\lambda}\left[\frac{\alpha\lambda}{e^{\alpha x}-1} + \frac{b}{x}\right]\right] \quad (40)$$

$$= (a)^n \prod_{i=1}^{n} x_i{}^b\, e^{-a\sum_{i=1}^{n} x_i{}^b[1-e^{-\alpha x_i}]^{\lambda}} \prod_{i=1}^{n}[1-e^{-\alpha x_i}]^{\lambda}\left[\frac{\alpha\lambda}{e^{\alpha x_i}-1} + \frac{b}{x_i}\right] \quad (41)$$

Take ln

$$l = n\ln a + \sum_{i=1}^{n}\ln x_i{}^b - a\sum_{i=1}^{n} x_i{}^b\,[1-e^{-\alpha x_i}]^{\lambda} + \lambda\sum_{i=1}^{n}\ln[1-e^{-\alpha x_i}] + \sum_{i=1}^{n}\ln\left[\frac{\alpha\lambda}{e^{\alpha x_i}-1} + \frac{b}{x_i}\right]$$

(42)

$$\frac{\partial l}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} x_i{}^b\,(1-e^{-\alpha x_i})^{\lambda} = 0 \quad (43)$$

$$\frac{\partial l}{\partial b} = \sum_{i=1}^{n}\ln x_i - a\sum_{i=1}^{n} x_i{}^b\,(1-e^{-\alpha x})^{\lambda}\ln x_i + \sum_{i=1}^{n}\frac{(e^{\alpha x_i}-1)}{\alpha\lambda x_i + be^{\alpha x_i} - b} = 0 \quad (44)$$

$$\frac{\partial l}{\partial \alpha} = -a\sum_{i=1}^{n} x_i{}^{b+1}\lambda[1-e^{-\alpha x_i}]^{\lambda-1}\,e^{-\alpha x_i} + \sum_{i=1}^{n}\lambda\left(\frac{x_i e^{-x_i}}{1-e^{-\alpha x_i}}\right) + \sum_{i=1}^{n}\frac{(\lambda(e^{\alpha x_i}-1) - \alpha\lambda x_i e^{\alpha x_i})}{\left(\frac{\alpha\lambda}{e^{\alpha x_i}-1} + \frac{b}{x_i}\right).(e^{\alpha x_i}-1)^2}$$

(45)

$$\frac{\partial l}{\partial \lambda} = -a\sum_{i=1}^{n} x_i^{\,b+1}\left(\ln(1-e^{-\alpha x_i})\right)(1-e^{-\alpha x_i})^{\lambda} + \sum_{i=1}^{n}\ln[1-e^{-\alpha x_i}] + \sum_{i=1}^{n}\frac{\alpha x_i}{\alpha\lambda x_i + b e^{\alpha x_i} - b} = 0$$

(46)

We can obtain the "MLES" of the parameters $\alpha, \lambda, a,$ and b by solving the equal. (43)-(46) numerically for $\alpha, \lambda, a,$ and b .

The method of moment estimator

Let $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \ldots\ldots, \mathbf{x_n}$ be random sample from the $\mathbf{X \sim W - EE(\alpha, \lambda, a, b)}$ the method of moment of EE-W distribution is defined by the following equation

$$E(X^r) = \sum_{s=1}^{n}\frac{1}{n}X_s^{\,r} \qquad (47)$$

Where $E(X^r)$ is the r^{th} moment about origin given equation (23).

For the case $r = 1$, equation (47) becomes as follows:

$$E(X^1) = a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(2+b(j+1))}{(\alpha n+\alpha)^{2+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(1+b(j+1))}{(\alpha m)^{1+b(j+1)}} = \sum_{s=1}^{n}\frac{1}{n}X_s^{\,1} = \overline{X} \qquad (48)$$

For the case $r = 2$, equation (47) becomes as follows:

$$E(X^2) = a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(3+b(j+1))}{(\alpha n+\alpha)^{3+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(2+b(j+1))}{(\alpha m)^{2+b(j+1)}} = \sum_{s=1}^{n}\frac{1}{n}X_s^{\,2} \qquad (49)$$

For the case $r = 3$, equation (47) becomes as follows:

$$E(X^3) = a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(4+b(j+1))}{(\alpha n+\alpha)^{4+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(3+b(j+1))}{(\alpha m)^{3+b(j+1)}}$$

(50)

For the case $r = 4$, equation (47) becomes as follows:

$$E(X^4) = a\alpha\lambda\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n}a^j}{j!}\binom{\lambda(j+1)-1}{n}\frac{\Gamma(5+b(j+1))}{(\alpha n+\alpha)^{5+b(j+1)}} + ab\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}a^j}{j!}\binom{\lambda(j+1)}{m}\frac{\Gamma(4+b(j+1))}{(\alpha m)^{4+b(j+1)}}$$

(51)

We can obtain the estimates for the parameters α, λ, a and b by solving the equations (48) _(51)for α, λ, a and b by using the numerical methods such as Newton Raphson method .

Applications

In this part , we explain the real data to show the importance of the $W - EE$ distribution , so that we will contrast the $W - EE(X; a, b, \alpha, \lambda)$ distribution with the following distributions

Exponentiated-Weibull $(E - W)$ with cdf $F(X; \alpha, \beta, \theta) = \left[1 - e^{(-\lambda x)^{\theta}}\right]^{\alpha}$, $x > 0$.

Odd Generalized – Exponential $(OGE - E)$ with cdf $F(X; a, \alpha, \lambda) = \left[1 - e^{-\lambda(e^{ax}-1)}\right]^{\alpha}$, $x > 0$.

Flexible – Weibull $(F-W)$ with cdf $F(X;\alpha,\lambda,\beta,\theta)=\left[1-e^{-e^{(\beta x^{\lambda}+\theta x^{\alpha})}}\right]$ $, x>0$.

In the order the $W-EE$ distribution with the above distributions the measures of good ness of fit including the Akaike Information criterion "AIC", Hannan-Quinn Information Criterion "HQIC" , Consistent Akaike Information criterion "CAIC" , and Bayesian Information criterion "BIC"are used [7] where :

$$AIC=-2\hat{\ell}+2q \quad , \quad BIC=-2\hat{\ell}+q\log(n)$$

$$CAIC=-2\hat{\ell}+\frac{2qn}{n-q-1} \quad , \quad HQIC=-2\hat{\ell}+2q\ \log(\log(n))$$

Where $\hat{\ell}$ denotes the logarithm likelihood function evaluated at the max likelihood estimates , "n" is the sample volume and "q" is number of parameters . In general , the distribution , which gives smallest values from criteria , shows the more suitable to the data .

Data Set

We have a real dataset corresponding to healing time (in weeks) of random sample of (128) Lung cancer patients **.**

Dataset are : 13.11,2.09,3.48,4.87,6.94,8.66,0.08,23.63,0.20,2.23,3.52,4.98,6.97,9.02,13.29,0.40,2.29,3.57,5.06,7.09,9.22,13.80,25.74,0.50,2.46,3.64,5.09,7.26,9.47,14.24,25.82,0.51,2.54,3.70,5.17,7.28,9.74,14.76,26.31,0.81,2.62,3.82,5.32,7.32,10.06,14.77,32.15,2.64,3.88,5.32,7.39,10.34,14.83,34,26,0.90,2.26,4.18,5.34,7.59,10.66,15.96,7.66,1.05,2.69,4.23,5.41,7.62,10.75,16.62,43.01,1.19,2.75,4.26,5.41,7.63,17.12,46.12,1.26,2.02,4.40,5.49,36.66,11.25,17.14,79.05,1.76,2.87,5.62,7.87,11.64,17.36,1.40,3.02,4.34,5.71,7.93,11.79,18.10,1.46,4.33,5.85,8.26,11.98,19.13,1.35,3.25,4.50,6.25,8.37,12.02,2.83,3.31,4.51,6.54,8.53,12.03,20.28,2.02,3.36,6.76,12.07,21.73,2.07,3.36,6.93,8.65,12.63,22.69 .

The "MLEs" of the type parameters for the data are given in chart (1.1) and the numerical values of the type selection statistics $\hat{l}$, "AIC" , "HQIC , " "CAIC" and "BIC" are listed in chart (1.2) We can see from chart (1.2) that **W − EE** type gives the smallest values for the criteria "AIC , HQIC , CAIC "and "BIC" so it represents the dataset better than the other selected types .

Chart 1.1.parameters estimates for the data

Type	Parameters estimates			
$W-EE(X;a,b,\alpha,\lambda)$	$\hat{a}=0.111$	$\hat{b}=1.015$	$\hat{\alpha}=0.413$	$\hat{\lambda}=0.436$
$E-W(X;a,\alpha,\lambda)$	$\hat{\alpha}=1.422$	$\hat{\lambda}=0.192$	$\hat{\theta}=0.79$	_
$OGE-E(X;a,\alpha,\lambda)$	$\hat{a}=0.1$	$\hat{\alpha}=0.1$	$\hat{\lambda}=0.1$	_
$F-W\ F(X;\alpha,\lambda,\beta,\theta)$	$\hat{\alpha}=0.1$	$\hat{\beta}=0.1$	$\hat{\lambda}=0.1$	$\hat{\theta}=0.539$

Chart 1. 2. The statistics $\hat{\ell}$, AIC ,HQIC , CAIC and "BIC"for the dataset .

Type	$\hat{\ell}$	AIC	HQIC	CAIC	"BIC"

$W-EE(X;a,b,\alpha,\lambda)$	-415.2734	838.5469	849.9550	838.8721	843.1820
$E-W(X;a,\alpha,\lambda)$	-420.655	847.2665	860.6747	847.4601	850.7429
$OGE-E(X;a,\alpha,\lambda)$	-7935505	1593.1	1606.5	1593.3	1596.6
$F-W\,F(X;\alpha,\lambda,\beta,\theta)$	-739.4106	1486.8	1498.2	1487.1	1491.5

References

[1] Mudholkar, Govind S., and Deo Kumar Srivastava. "Exponentiated Weibull family for analyzing bathtub failure-rate data." IEEE transactions on reliability 42.2 (1993): 299-302.

[2] Mudholkar, Govind S., Deo Kumar Srivastava, and Marshall Freimer. "The exponentiated Weibull family: A reanalysis of the bus-motor-failure data." Technometrics 37.4 (1995): 436-445.

[3] Tahir, M. H., et al. "A new Weibull-G family of distributions " Hacettepe Journal of Mathematics and Statistics 45.2 (2016): 629-647.

[4] Pal, Manisha, M. Masoom Ali, and Jungsoo Woo. "Exponentiated weibull distribution." Statistica 66.2 (2006): 139-147.

[5] Merovci, Faton, and Ibrahim Elbatal. "Weibull-Rayleigh distribution: theory and applications." Appl. Math. Inf. Sci 9.5 (2015): 1-11.

[6] Luguterah, Albert. "Odd generalized exponential Rayleigh distribution." Advances and Applications in Statistics 48.1 (2016): 33-48.

[7] ELGARHY, Mohammed, and Amal HASSAN. "Exponentiated Weibull Weibull Distribution: Statistical Properties and Applications." Gazi University Journal of Science 32.2 (2019): 616-635.

[8] Eissa, Fathy Helmy. "The exponentiated Kumaraswamy–Weibull distribution with application to real data." International Journal of Statistics and Probability 6.6 (2017): 167-82.

[9] Aban, Inmaculada B., Mark M. Meerschaert, and Anna K. Panorska. "Parameter estimation for the truncated Pareto distribution." Journal of the American Statistical Association 101.473 (2006): 270-277.

[10] Sabah Muhammad, and Kareema Abad Al-Kadim. " Odd Generalized Exponential Weibull Exponential Distribution." Journal of Engineering and Applied Sciences. 14. 8(2019).10360-10368.

[11] Zeadan Khalaf, Reyah, and Kareema Abad Al-Kadim. "Truncated Rayleigh Pareto Distribution." Journal of Physics Conference Series. Vol. 1591. No. 1. 2020.