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Analytical Solutions Of Some Special Nonlinear Partial Differential Equations Using Aboodh-Adomian Decomposition Method

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ABSTRACT

We apply the Aboodh-Adomian Decomposition Method (AADM) in this study to solve nonlinear Benjamin-Bona-Mahony (BBM) and Fisher's partial differential equations (PDE). This method, being an integral transform, is a hybrid of two well-known and efficient methods: the Aboodh transform and the Adomian decomposition method. The method is demonstrated by solving two special cases of the BBM Equation and one special case of Fisher's partial differential equation. Because of its high convergence rate in approximating exact solutions, this approach is very dependable. The method can also produce numerical solutions without the usage of restrictive assumptions or the discretization typical of numerical methods; making it free of round-off errors. The Aboodh-Adomian Decomposition method employs a straightforward computation that leads to effectiveness. The efficiency of AADM is demonstrated in the significant reduction of number of numerical computations. The effectiveness and efficiency of EADM account for its broad application, particularly for higher order PDEs.

Keywords: Aboodh Transform, Adomian Polynomials, PDEs, Analytical Solutions, Benjamin-Bona-Mahony Equation, Fisher's Equation

INTRODUCTION

Nonlinear differential equations are incredibly essential to humans since most physical phenomena are nonlinear in nature and are modelled by these equations. In this regard, partial differential equations (PDEs) are in particular, fundamental. Unfortunately, analytical techniques cannot be used to solve the majority of nonlinear problems. In addition, to solve nonlinear problems, standard numerical methods require perturbation, discretization, linearization, or transformation. However, Adomian (1994) established that the Adomian decomposition method is free of such steps as are involved in standard numerical methods and is thus widely used in the literature. Several researchers have made significant efforts and implemented diverse methods for solving nonlinear PDEs over the last few decades. Recently Ali et al., (2018) applied the Laplace Adomian decomposition method in finding the approximate solutions for nonlinear general fisher's equation. In a similar vein, we reference the works of (Khuri, 2001; Wazwaz, 2010; Wazwaz and Mehanna, 2010).

Another important method that has received little attention is the Aboodh-Adomian decomposition method (AADM). It was introduced by Ige, et al., (2022). It is a combination of the Aboodh transform and the Adomian decomposition method, two well-known and efficient methods. It is possible to obtain numerical solutions using this method without the use of restrictive assumptions or discretization general fisher's equation. In a similar vein, we reference the works of (Khuri, 2001; Wazwaz, 2010; Wazwaz and Mehanna, 2010).

making it free of round-off errors. A solution in the form of a finite series is also achieved using this method, and it has the highest and fastest rate of convergence. In this paper we apply EADM to obtain the analytical solution of some special nonlinear partial differential equations: The Benjamin-Bona-Mahony (BBM) Equations and Fisher's Equations.

The Benjamin-Bona-Mahony (BBM) Equations

The partial differential equation Benjamin–Bona–Mahony (BBM), commonly known as the regularized long-Wave equation (RLWE), was introduced by Benjamin et al. (1972). (See also Muhammad et al., 2019).

$U_t+U_x+UU_x-U_{xxt} = 0, \quad (x,0) = f(x) \quad (1.1)$

Benjamin, Bona, and Mahony explored this equation in 1972 as an improvement on the Korteweg de Vries equation (KdV equation) for modeling long surface gravity waves of small amplitude traveling in 1+1 dimensions. They demonstrated the BBM equation's solutions' stability and uniqueness. The KdV equation, on the other hand, is unstable in its high wavenumber components. Furthermore, the KdV equation has an infinite number of motion integrals, whereas the BBM equation has just three (Molati and Khalique, 2012)

In physical applications, the BBM equation is well-known. It offers a model for long-wave propagation that includes nonlinear and dissipative phenomena. It's used to study long-wavelength surface waves in liquids, cold plasma hydro magnetic waves, compressible fluids acoustic-gravity waves, and harmonic crystal acoustic waves (Molati and Khalique, 2012). The dynamics of the BBM equation has drawn the attention of many mathematicians (Singh et al., 2011). For shallow water waves, the BBM equation has been examined as a regularized version of the Kdv equation. Finding analytic solutions to the nonlinear BBM Equation is crucial, as the equation also models complicated physical systems that can occur in engineering, chemistry, biology, mechanics, and physics (Talha and Khaled, 2009).

Fisher's Equations

As a nonlinear model for a physical system comprising linear diffusion and nonlinear growth, the Fisher equation assumes the following non-dimensional form:

$Ut= \quad Uxx+(1-U^{\alpha})(U-\rho), \quad U(x,0) \quad =g(x)$ (1.2)

A constant-velocity front of transition from one homogeneous condition to another is described by (1.2) kink-like traveling wave solutions called Solitons. Solitons, on the other hand, emerge as a result of a delicate balancing between weak nonlinearity and dispersion. As a result, in Mathematics and Physics, a soliton is defined as a self-reinforcing solitary wave—a wave packet or pulse that keeps its shape while traveling at steady velocity. The dispersion relation between the frequency and the speed of the waves is referred to as "dispersive effects." Solitons are solutions to a class of weakly nonlinear dispersive partial differential equations that describe physical systems. Instead of dispersion, when diffusion occurs, energy released by nonlinearity balances energy consumed by diffusion, resulting in moving waves or fronts.

As a consequence, moving wave fronts are a well-studied solution form for reaction diffusion equations, with applications in chemistry, biology, and medicine (Wazwaz and Gorguis, 2004).

Description of Aboodh **Transform**

Aboodh transform is an integral transformation defined for function of exponential order (Aboodh 2022). Consider the function in the set A defined as;

$A= \{(t): \exists M, c1, c2 >0, |(t)|< M\, e^{|t|ci}\ , \ if\ t \in (-1)^{i}\times [0,\infty\ )\}$ (2.1)

Where for any given function in the set A defined above, the constant $c1,2$ may be either finite or infinite, but M must be infinite. According to Aboodh (2022), Aboodh Transform is defined as:

$A\ \{(t)\} = \frac{1}{W^2}\int_0^{\infty} f(\frac{t}{w})e^{-t}\ dt=K(w), t\geq 0, w\in(c1, c2)$ (2.2)

Or

A $\{(t)\} = \frac{1}{w}\int_0^{\infty} f(t)e^{-tw}dt = K(w),\ t\geq 0,\ w \in (c1, c2)$ (2.3)

We note here, that w in the above definition is used to factor t in the analysis of function f.

Aboodh Transform of Partial Derivatives Aboodh et al. (2022) extended the method to solving partial differential equations. The Aboodh transform of partial derivatives are obtained through integration by parts, then we find the following expressions

$$A\left[\frac{\partial f(x,t)}{\partial t}\right] = w\,K(x,w) - \frac{1}{w}f(x,0) \quad (2.4)$$

$$A\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = w^2\,K(x,w) - f(x,0) - \frac{1}{w}\,\frac{\partial f(x,0)}{\partial t} \quad (2.5)$$

$$A\left[\frac{\partial^3 f(x,t)}{\partial t^3}\right] = w^3\,K(x,w) - wf(x,0) - \frac{\partial f(x,0)}{\partial t} \quad (2.6)$$

$$A\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{d}{dx}[K(x,w)] \quad (2.7)$$

$$A\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{d^2}{dx^2}[K(x,w)] \quad (2.8)$$

$$A\left[\frac{\partial^3 f(x,t)}{\partial t^3}\right] = \frac{d^3}{dx^3}[K(x,w)] \quad (2.9)$$

Aboodh Transform of Some Functions

By using the definition of Aboodh transform of equations (2.2)-(2.3) on some functions the results can be generated as tabulated in table 1. (Aboodh, 2022).

Table 1: Table of Functions and their Aboodh Transform

$\mathbf{f(t)}$	$\mathbf{A[f(t)] = F(w)}$
1	$\frac{\mathbf{1}}{\mathbf{w^2}}$
t	$\frac{\mathbf{1}}{\mathbf{w^3}}$
t^2	$\frac{2!}{\mathbf{w^4}}$
$t^n \quad n \in N$	$\frac{n!}{\mathbf{w^{n+2}}}$
e^{at}	$\frac{\mathbf{1}}{\mathbf{w^2 - a}}$
$\sin(at)$	$\frac{\mathbf{a}}{\mathbf{w^3 + a^2 w}}$
$\cos(at)$	$\frac{\mathbf{1}}{\mathbf{w^2 + a^2}}$

METHODOLOGY

In this paper, our interest is to solve some special nonlinear partial differential equations which are third order Benjamin-Bona-Mahony and second order Fisher's equations. We first demonstrate how the Aboodh transform method can be used to decompose the general nonlinear partial differential equation. (2018), we consider;

$$\frac{\partial^n u(x,t)}{\partial t^n} + Ru(x,t) + Nu(x,t) = g(x,t) \qquad (3.1)$$

Where n=1, 2, 3…

And the initial condition is given as

$$\frac{\partial^{n-1} u(x,t)}{\partial t^{n-1}} |_{t=0} = f_{n-1}(x)$$

where $\frac{\partial^n u(x,t)}{\partial t^n}$ is the partial derivative of function $u(x,t)$ of nth order, while R represents the linear differential operator, $Nu(x,t)$ represents the nonlinear terms of the differential equations, and $f(x,t)$ indicates the non-homogeneous (source) term.

Applying the Aboodh transform on equation (3.1) we have;

$$A\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] + A[Ru(x,t)] = A[g(x,t)] \qquad (3.2)$$

We recall that

$$A\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = w^n A[u(x,t)] - \sum_{k=0}^{n-1} \frac{1}{w^{2-n+k}} \frac{\partial^k u(x,0)}{\partial t^k} \qquad (3.3)$$

Substituting Equation (3.3) into Equation (3.2), we have;

$$A[u(x,t)] = \frac{1}{w^n} A[g(x,t)] + \sum_{k=0}^{n-1} \frac{1}{w^{2+k}} \frac{\partial^k u(x,0)}{\partial t^k} - \frac{1}{w^n}\ A[Ru(x,t)] + A[Nu(x,t)] \qquad (3.3)$$

Applying the inverse Aboodh transform to Equation (3.3), we have;

$$u(x,t) = A^{-1}\left[\frac{1}{w^n} A[g(x,t)] + \sum_{k=0}^{n-1} \frac{1}{w^{2+k}} \frac{\partial^k u(x,0)}{\partial t^k}\right] - A^{-1}\left[\frac{1}{w^n}\ A[Ru(x,t)] + A[Nu(x,t)]\right] \ (3.4)$$

We can rewrite this as;

$$u(x,t) = F(x,t) - A^{-1}\left[\frac{1}{w^n}\ A[Ru(x,t)] + A[Nu(x,t)]\right] \qquad (3.5)$$

Where $(x,)$ represents the expression that rises from the given initial conditions and the source terms after simplification. We note here that, our solution will be in the form of infinite series as

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \qquad (3.6)$$

We can now decompose the nonlinear term as

$$Nu(x,t) = \sum_{k=0}^{\infty} A_k \qquad (3.7)$$

Where A_k is defined as the Adomian polynomials which can be generated using the formula

$$A_k = \frac{1}{K!} \frac{\partial^k}{\partial y^k}\left[N\left(\sum_{j=0}^{\infty} \rho^j u_j\right)\right] \rho = 0, k = 0,1.., \qquad (3.8a)$$

Where ρ is taken as formal parameter and we will drop it after the calculation by equating it to zero.

According to Aboodh (2011), Adomian polynomial can be computed in different ways it is not unique and we can calculate it from the Tylor expansion of function (u) around the first component u_0 i.e.

$(u)= \sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty} \frac{(u-u_0)^0}{k!} f^{(k)}(u_0)$ (3.8b)

Here, we provided the first five Adomian polynomials for the

Nonlinear terms $Nu=(u)$

$A_0=(u_0)$

$A_1=u_1 f'(u_0)$,

$A_2= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0)$,

$A_3=u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0)$,

$A_4=u_4 f'(u_0) + (u_1 u_2 + \frac{u_2^2}{2!}) f''(u_0) + (\frac{u_1^2 u_2}{2!}) f'''(u_0) + \frac{u_4^2}{4!} f^{(4)}(u_0)$,

Now substituting Equation (3.7) and Equation (3.6) into Equation (3.5) we have that;

$$\sum_{k=0}^{\infty} u_k(x,t) = F(x,t) - A^{-1}\left[\frac{1}{w^n} A[R \sum_{k=0}^{\infty} u_k(x,t)] + A[\sum_{k=0}^{\infty} A_k]\right] \quad (3.9)$$

Then starting to evaluate from equation (3.9) at k=0, we have $u_0(x,t)=F(x,t)$ (3.10)

And the recursive relation from equation (3.9) given as:

$$u_{k+1}(x,t) = -A^{-1}\left[\frac{1}{w^n} A[Ru_k(x,t)] + A[(A_k)]\right] \quad (3.11)$$

Where n=1, 2, 3 (from the order of the PDE) and $k\geq 0$. The analytical solution $(x,)$ can be approximated by truncated series.

$$u(x,t)= \lim_{k\to\infty} \sum_{k=0}^{\infty} u_k(x,t) \quad (3.12)$$

The infinite series in equation (3.12) may converge completely very fast to exact solution or with few terms truncation will result to the exact solution of the given differential equation.

NUMERICAL IMPLEMENTATIONS

$A[U(x,t)] = \frac{\mathbf{1}}{\mathbf{w^2}} (x,0)+ \frac{\mathbf{1}}{\mathbf{w}} A[U_{xxt}-U_x-UU_x]$ (4.3)

Introducing the initial condition and taking the inverse of the Aboodh transform we have;

$(x,t)=x+A^{-1}[\frac{\mathbf{1}}{\mathbf{w}} A[U_{xxt}-U_x-UU_x]]$ (4.4)

But

Numerical Problem 1

We consider the BBM equation, $U_t+U_x+UU_x-U_{xxt}=0$, $(x,0)=x$ (4.1)

Taking Aboodh transform of each term we have;

$A[U_t] = A[U_{xxt}]-A[U_x+UU_x]$ (4.2)

Where

$A[U_t] = w A[(x,)] - \frac{1}{w} U(x,0)$

Thus Equation (4.2) becomes;

$$U(x,t) = \sum_{k=0}^{\infty} U_k(x,t) \quad (4.5)$$

And

$$UU_x = \sum_{k=0}^{\infty} A_k \quad (4.6)$$

Substituting Equations (4.5)-(4.6) in Equation (4.4) we have

$U_0=x$ (4.7)

and the recurrence relation as

$U_{k+1}=\mathrm{A}^{-1}[\frac{1}{\mathrm{w}}\mathrm{A}$ $[U_{kxxt}-U_{kx}-A_k]]$
(4.8)

Now we compute the individual terms from the recurrence equation. $U_1=\mathrm{A}^{-1}[\frac{1}{\mathrm{w}}\mathrm{A}\ [U_{0xxt}-U_{0x}-A_0]]$
(4.9)

Where,

$A_0=U_0U_{0x}=x\cdot 1=$, $U_{0xxt}=0$ and $U_{0x}=1$

Thus Equation (4.9) becomes

$U_1=\mathrm{A}^{-1}\ [\ \frac{1}{\mathrm{w}}\mathrm{A}\ [0-1-x]]$

$=\ \mathrm{A}^{-1}[\ \frac{1}{\mathrm{w}^3}(-1-x)]$

$U_1=$ $-$ $(1+x)$ t
(4.10)

Again from equation (4.8) we have;

$U_2=\mathrm{A}^{-1}$ $[\frac{1}{\mathrm{w}}\mathrm{A}$ $[U_{1xxt}-U_{1x}-A_1]]$
(4.11)

Where,

$A_1=U_0\frac{\partial \mathrm{U}_1}{\partial \mathrm{x}}+U_1\ \frac{\partial \mathrm{U}_0}{\partial \mathrm{x}}$

$=(-t)-(1+x)t\cdot 1=-(2x+1)\mathrm{t}$

Thus, it follows from equation (4.11) that

U_2 $=$ $(x+1)$ t^2
(4.12)

Similarly, from equation (4.8) we compute $U3$ as follows;

$U_3=\mathrm{A}^{-1}$ $[\frac{1}{\mathrm{w}}\mathrm{A}[U_{2xxt}-U_{2x}-A_2]]$
(4.13)

Where;

$A_2=U_0\frac{\partial \mathrm{U}_2}{\partial \mathrm{x}}+U_1\ \frac{\partial \mathrm{U}_1}{\partial \mathrm{x}}+\mathrm{U2}\ \frac{\partial \mathrm{U}_0}{\partial \mathrm{x}}$

$=(3x+2)\ t^2$

Hence equation (4.13) becomes;

$U3=\frac{2!}{3!}.\left(-3(\mathrm{x}+1)\right).\mathrm{A}^{-1}\left[\frac{3!}{\mathrm{W}^5}\right]$

$U_3=$ $-$ $(x+1)$ t^3
(4.14)

Therefore, the solution of equation (4.1) is

U $(x,)=U_0+U_1+U_2+U_3+\cdots$
(4.15)

$=\mathrm{x}+\left[\frac{-(1+\mathrm{x})\mathrm{t}}{1+\mathrm{t}}\right]$

U $(,)$
$=$
$\frac{\mathrm{x}-\mathrm{t}}{1+\mathrm{t}}$
(4.16)

CONCLUSION

In this paper, the Aboodh -Adomian Decomposition Method (AADM) has been successfully applied to find the solutions of nonlinear Benjamin-Bona-Mahony and Fisher's equations as presented in figures 1-5. It is observed that the use of hybrid EADM provides very good approximate solutions when compared with exact values than Adomian Decomposition Method (ADM). The method transforms these equations to recurrences relation whose terms can be computed with the aid of any symbolic computational environment such as Maple, Mathematica, and Scientific workplace among others. The solution using this method is usually in the form of a finite series and it has high and fastest rate of convergence to the exact solutions of the relevant problems. It is possible to obtain numerical solutions using this method without the use of restrictive assumptions or discretization, making it free of round-off errors. The Aboodh –Adomian Decomposition method use a simple and straightforward calculation. The number of numerical computations is decreased. The efficiency of AADM and the reduction in calculations demonstrate its extensive applicability, particularly for higher order PDEs.

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