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The Odd Truncated Inverse Exponential- Weibull -Exponential Distribution

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Abstract

There are many different continuous lifetime distributions that can be used to model lifetime data in many fields of the real life. In our life, there are many important problems where the real data do not fit any of the known lifetime distributions, so we need to improve these lifetime distributions to be more flexible for real data sets. We introduce new continuous distribution called the odd truncated inverse exponential Weibull- exponential distribution (OTIE-W-ED) for modeling life time data.

Keywords: Distribution, Function, Quantile, Median, Moments, Mean, Variance, The coefficient of skewness, Kurtosis, Variation, The moment generating functions, and order statistics.

INTRODUCTION

In scientific field we need to develop a new distributions, so we can use it in a wide range of fields like reliability analysis. We will introduce a new distribution called the odd truncated inverse exponential – Weibull-exponential distribution(OTIE-W-ED) using methods like the one that used by Luguterah A. and Nasiru The Odd Generalized Exponential General Linear Exponential (OGE-GLE) distribution [6]. The odd generalized exponential family[4,14]. In this paper, we introduce The Odd Truncated Inverse Exponential-Weibull-Exponential Distribution (OTIE-W-D) by replacing x in the cdf of the truncated inverse exponential distribution] with (F(x))/(S(x)) where(F(x)is the cdf of Weibull Distribution and S(x) is the survival function of the Exponential distribution and study some of its important functions like reliability function ,hazard ,reverse hazard ,cumulative hazard function and some statistical and mathematical properties such as quantile function and median, the mean, rth moment about the origin, rth moment about the mean. Coefficient of skewness, kurtosis and variation, the moment generating function and order statistics the parameter of the (OTIE-W-D) distribution are estimated by using the maximum likelihood estimation method . We illustrate the importance of the new family by applications to a real data set and we will compare our proposed distributions with some other distributions by using criteria like the AIC, CAIC, HQIC and BIC. In this study the shapes of the functions of our new distribution and the comparisons in the given tables are introduced by MATLAB (R2012b) software.

CDF AND PDF OF OTIE-W-ED

We studied the Odd Truncated Inverse Exponential- Weibull-Exponential Distribution by using the CDF of Weibull Distribution as follows:

$$F(x) = 1 - e^{-ax^b} \quad , \quad x > 0 \qquad (1)$$

And the reliability function of Exponential Distribution as follows:

$$S(x) = e^{-kx} \quad , x > 0 \qquad (2)$$

By substituting x of the CDF of the truncated inverse exponential distribution:

$$G(x) = \frac{e^{\frac{-\alpha}{x}}}{e^{\frac{-\alpha}{x_\circ}}} \qquad (3)$$

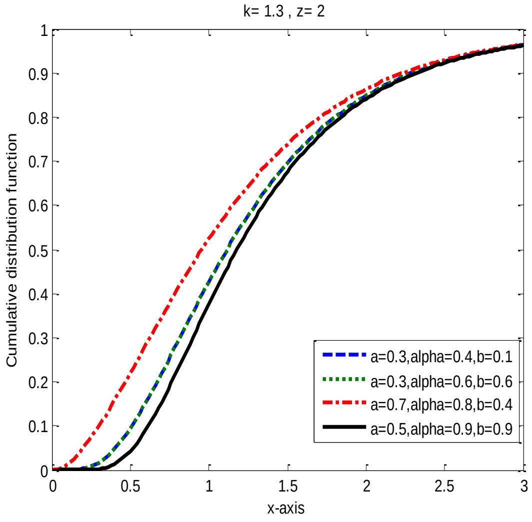

Figure 1: *The Cumulative Distribution Function*

From Figure 1, we note that the CDF increases as the value of a ,α ,b decreases

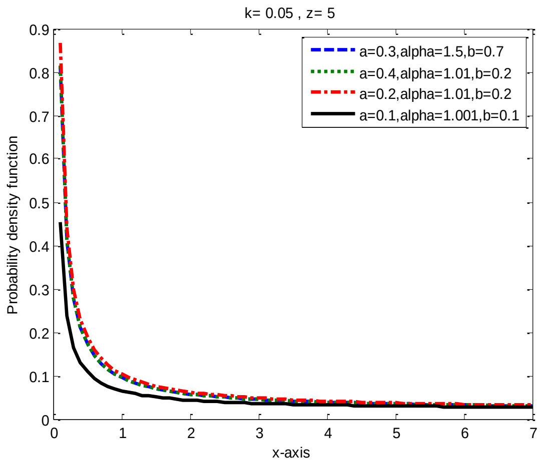

Figure 2 : *The Probability Density Function*

From Figure 2 , we note that the PDF is increases as the value a ,α ,b decreases

By using $x = F(x)/S(x)$

We get the CDF of the new distribution called the odd truncated inverse exponential- Weibull-exponential distribution

$$\boldsymbol{G}(x) = \frac{e^{-\frac{\alpha}{\frac{F(x)}{S(x)}}}}{e^{\frac{-\alpha}{x_\circ}}} = \frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}}{e^{\frac{-\alpha}{x_\circ}}} \quad [\text{CDF of the OTIE} - \text{W} - \text{ED}] \qquad (4)$$

The PDF of the OTIE-W-ED is given by :

$$g(x) = \acute{G}(x) =$$

$$\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}}{e^{\frac{-\alpha}{x_\circ}}}\left[-\alpha\, e^{-kx}.-abx^{b-1}e^{-ax^b}(1 - e^{-ax^b})^{-2} + (1 - e^{-ax^b})^{-1}.\,\alpha\, ke^{-kx}\right] \quad a ,b ,k ,\alpha > 0 \text{ and } 0 < x < x_\circ$$

$$g(x) = \frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}.\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}\left(1-e^{-ax^b}\right)}\left[k + \frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right] \qquad (5)$$

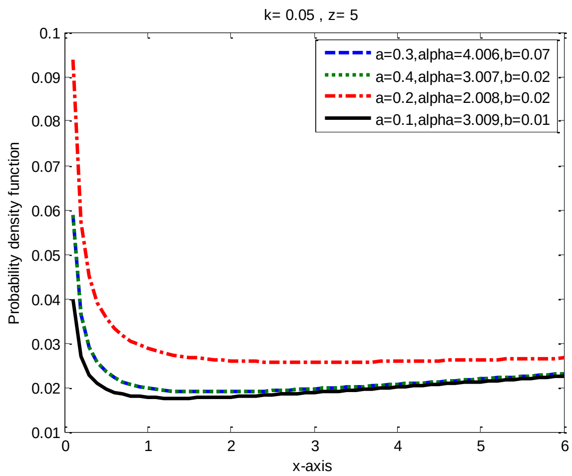

Figure 3: *The Probability Density Function*

From Figure 3, we note that the PDF is increases as the value a ,α ,b decreases

THE LIMIT OF THE CDF AND THE PDF OF THE OTIE-W-ED

$$\lim_{x\to 0} G(x) = \lim_{x\to 0}\frac{e^{\frac{-\alpha\ e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}} = \frac{e^{-\frac{\alpha}{0}}}{e^{-\frac{\alpha}{x_\circ}}} = \frac{e^{-\infty}}{e^{-\frac{\alpha}{x_\circ}}} = \frac{0}{e^{-\frac{\alpha}{x_\circ}}} = 0 \qquad (6)$$

$$\lim_{x\to x_\circ} G(x) = \lim_{x\to x_\circ} \frac{e^{\frac{-\alpha\, e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}} = \frac{e^{\frac{-\alpha\, e^{-kx_\circ}}{1-e^{-ax_\circ^b}}}}{e^{-\frac{\alpha}{x_\circ}}} > 0 \qquad (7)$$

Since $(0, x_\circ) \subset (0, \infty)$ and $\lim_{x\to x_\circ} G(x) = 1$ when we know that $\frac{1-e^{-ax^b}}{e^{-kx}} \cong x_\circ$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} . \alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}(1-e^{-ax^b})} \left[k + \frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right] = \frac{0}{e^{-\frac{\alpha}{x_\circ}}} = 0 \quad (8)$$

$$\lim_{x\to x_\circ} g(x) = \lim_{x\to x_\circ} \left[\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} . \alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}(1-e^{-ax^b})} \left[k + \frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]\right] > 0 \qquad (9)$$

Since $\frac{1-e^{-ax^b}}{e^{-kx}} \cong x_\circ$

$$\frac{e^{-\alpha/x_\circ}.\alpha}{e^{\frac{-\alpha}{x_\circ}}(x_\circ)} \left[k + \frac{abx^{b-1}e^{-ax^b}}{x_\circ e^{-kx}}\right] = \frac{\alpha}{x_\circ} \left[k + \frac{abx^{b-1}e^{-ax^b}}{x_\circ e^{-kx}}\right]$$

$(0 < x < x_\circ)$.

RELIABILITY FUNCTIONS OF THE OTIE-W-ED

We will show some reliability functions of the OTIE-W-ED :

Reliability Function

$$R(x) = 1 - G(x)$$

$$R(x) = 1 - \frac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}} \qquad (10)$$

$$\lim_{x\to 0} R(x) = \lim_{x\to 0} \left(1 - \frac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}}\right) = 1 \qquad (11)$$

$$\lim_{x\to x_\circ} R(x) = \lim_{x\to x_\circ} \left(1 - \frac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}}\right) = 0 \qquad (12)$$

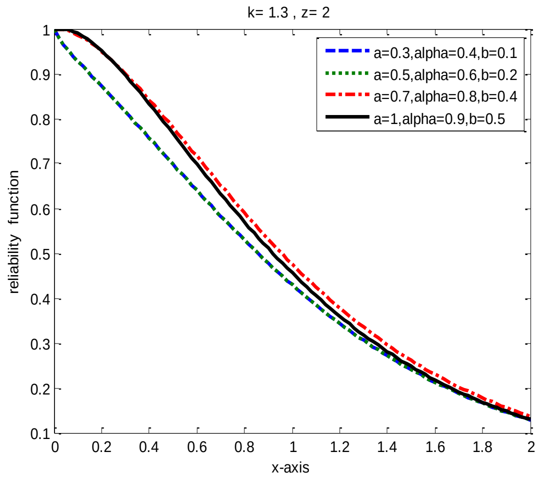

Figure 4 : *The Reliability Function*

From Figure 4 , we note that the reliability function is increases as the value a ,α ,b decreases

Hazard Function

$$H(x) = \frac{g(x)}{R(x)}$$

H(x)

$$= \frac{\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} . \alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}(1-e^{-ax^b})} \left[k + \frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]}{1 - \frac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}}} \qquad (13)$$

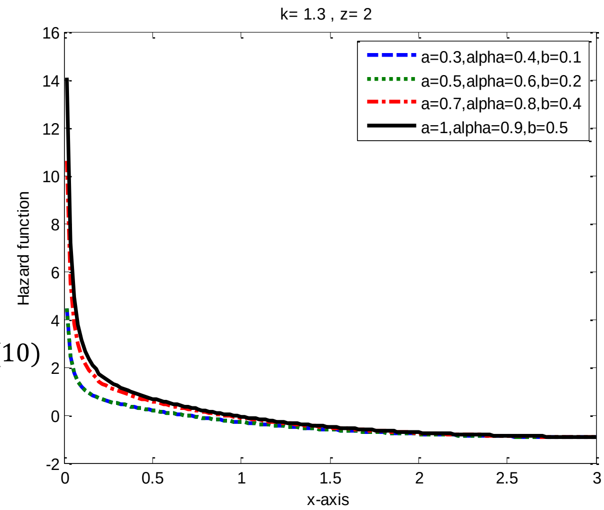

Figure 5: *The Hazard Function*

From Figure 5 , we note that the hazard function is increases as the value a ,α ,b increases .

Reverse Hazard Function

$$r(x) = \frac{g(x)}{G(x)}$$

$$r(x) = \frac{\dfrac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} .\,\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}(1-e^{-ax^b})}\left[k+\dfrac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]}{\dfrac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}}}$$

$$r(x) = \alpha\, e^{-kx}(1-e^{-ax^b})^{-1}\left[k+\frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right] \quad (14)$$

The Cumulative Hazard Function

$$H(x) = -\ln R(x) = -\ln\left[1-\frac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}}\right] \quad (15)$$

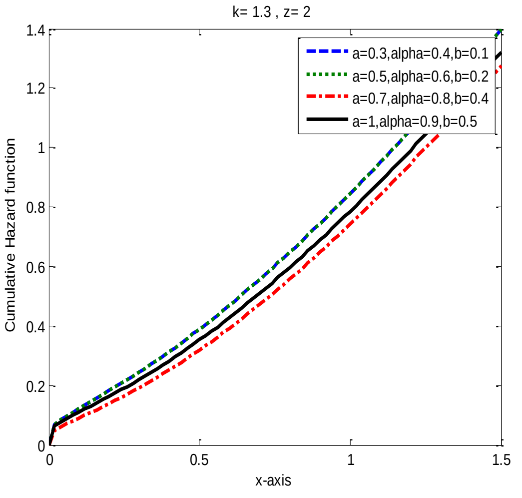

Figure 6: *The Cumulative Hazard Function*

From Figure 6, we note that the cumulative hazard function is increases as the value a ,α ,b decreases.

PROPERTIES OF THE OTIE-W-ED

Quintile and Median

The quintile can be written as:

$$G(x) = Q, \qquad 0 < Q < 1 \quad (16)$$

$$\frac{e^{-\frac{\alpha e^{-kx}}{1-e^{-ax^b}}}}{e^{-\frac{\alpha}{x_\circ}}} = Q \quad (17)$$

To find x we can use the numerical methods such as Newton Raphson method to solve this equation

$$-\frac{\alpha e^{-kx}}{1-e^{-ax^b}} = \ln Q - \frac{\alpha}{x_\circ} \quad (18)$$

To find the median of the OTIE-W-ED by setting Q=1/2 and solve the equation numerically .

THE MOMENTS AND COEFFICIENT OF SKEWNESS, KURTOSIS AND VARIATION

The r-th moment about the origin for the OTIE-W-ED is given by:

$$E(X^r) = \frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(an)^q k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!j!p!i!\binom{p+1}{n}(-1)^n}\left[\frac{x_\circ^{\,j+r+i+bq+1}}{j+r+i+bq+1}\right] + \frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(-a)^s(a)^{q+1}n^{\,q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^n q!j!p!i!s!}\left[\frac{x_\circ^{\,r+b+j+i+bs+bq}}{r+b+j+i+bs+bq}\right] \quad (19)$$

$$E(X^r) = \int_0^{x_\circ} x^r . \left[\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} . \alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}\left(1-e^{-ax^b}\right)} \left[k + \frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]\right] \qquad (20)$$

$$E(X^r) = \int_0^{x_\circ} [\frac{x^r k\alpha e^{\frac{\alpha}{x_\circ}}\ e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} .\ e^{-kx}}{\left(1-e^{-ax^b}\right)} + \frac{x^r ab\alpha x^{b-1} e^{\frac{\alpha}{x_\circ}}\ e^{-kx}e^{-ax^b}e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}}{(1-e^{-ax^b})^2}]dx$$

$$E(X^r) = k\alpha e^{\frac{\alpha}{x_\circ}} \int_0^{x_\circ} \frac{x^r e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} .\ e^{-kx}}{\left(1-e^{-ax^b}\right)} dx + ab\alpha e^{\frac{\alpha}{x_\circ}} \int_0^{x_\circ} \frac{x^{r+b-1}\ e^{-kx}e^{-ax^b}e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}}{(1-e^{-ax^b})^2} dx \qquad (21)$$

$$e^{-kx} = \sum_{j=1}^{\infty} \frac{(-kx)^j}{j!} \qquad (22)$$

From substituting Eq. (22) in Eq.(21) we get:

$$E(X^r) = \frac{\sum_{j=0}^{\infty}(-k)^j k\alpha e^{\frac{\alpha}{x_\circ}}}{j!} \int_0^{x_\circ} \frac{x^{r+j} e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}.}{\left(1-e^{-ax^b}\right)} dx + \frac{\sum_{j=0}^{\infty}(-k)^j ab\alpha e^{\frac{\alpha}{x_\circ}}}{j!} \int_0^{x_\circ} \frac{x^{r+b+j-1}\ e^{-ax^b}e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}}{(1-e^{-ax^b})^2} dx \qquad (23)$$

$$e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}} = \sum_{p=0}^{\infty} \frac{[-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}]^p}{p!} \qquad (24)$$

From substituting Eq (24) in Eq(23) we get:

$$E(X^r) = \frac{\sum_{p=0}^{\infty}\ \sum_{j=0}^{\infty}(-k)^j(-\alpha)^p k\alpha e^{\frac{\alpha}{x_\circ}}}{j!p!} \int_0^{x_\circ} \frac{x^{r+j}e^{-kpx}}{(1-e^{-ax^b})^{p+1}} dx + \frac{\sum_{p=0}^{\infty}\ \sum_{j=0}^{\infty}(-k)^j(-\alpha)^p ab\alpha e^{\frac{\alpha}{x_\circ}}}{j!p!} \int_0^{x_\circ} \frac{x^{r+b+j-1}\ e^{-ax^b}e^{-kpx}}{(1-e^{-ax^b})^{p+2}} dx \qquad (25)$$

$$e^{-kpx} = \sum_{i=0}^{\infty} \frac{(-kpx)^i}{i!} \qquad (26)$$

From substituting Eq (26) in Eq (25) we get:

$$E(X^r) = \frac{\sum_{i=0}^{\infty}\ \sum_{p=0}^{\infty}\ \sum_{j=0}^{\infty}(-k)^j(-\alpha)^p(-kp)^i\ k\alpha e^{\frac{\alpha}{x_\circ}}}{j!\,p!\,i!} \int_0^{x_\circ} \frac{x^{r+j+i}}{(1-e^{-ax^b})^{p+1}} dx$$

$$+ \frac{\sum_{i=0}^{\infty}\ \sum_{p=0}^{\infty}\ \sum_{j=0}^{\infty}(-k)^j(-\alpha)^p\ (-kp)^i ab\alpha e^{\frac{\alpha}{x_\circ}}}{j!\,p!\,i!} \int_0^{x_\circ} \frac{x^{r+b+j+i-1}\ e^{-ax^b}}{(1-e^{-ax^b})^{p+2}} dx \qquad (27)$$

$$e^{-ax^b} = \sum_{s=0}^{\infty} \frac{(-ax^b)^s}{s!} \qquad (28)$$

From substituting Eq (28) in Eq (27) we get:

$$E(X^r) = \frac{\sum_{i=0}^{\infty}\ \sum_{p=0}^{\infty}\ \sum_{j=0}^{\infty}(-k)^j(-\alpha)^p(-kp)^i\ k\alpha e^{\frac{\alpha}{x_\circ}}}{j!\,p!\,i!} \int_0^{x_\circ} \frac{x^{j+r+i}}{(1-e^{-ax^b})^{p+1}} dx$$

$$+ \frac{\sum_{s=0}^{\infty}\ \sum_{i=0}^{\infty}\ \sum_{p=0}^{\infty}\ \sum_{j=0}^{\infty}(-k)^j(-\alpha)^p\ (-kp)^i(-a)^s ab\alpha e^{\frac{\alpha}{x_\circ}}}{j!\,p!\,i!\,s!} \int_0^{x_\circ} \frac{x^{r+b+j+i+bs-1}.}{(1-e^{-ax^b})^{p+2}} dx \qquad (29)$$

$$(1-e^{-ax^b})^{p+1}=\sum_{n=0}^{p+1}\binom{p+1}{n}(-1)^n(e^{-ax^b})^n \tag{30}$$

$$(1-e^{-ax^b})^{p+2}=\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^n(e^{-ax^b})^n \tag{31}$$

From substituting Eq (30) and Eq (31) in Eq (29)

$$E(X^r)=\frac{\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^j(-\alpha)^p(-kp)^i\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} j!\,p!\,i!\binom{p+1}{n}(-1)^n}\int_0^{x_\circ}x^{j+r+i}e^{anx^b}\,dx$$
$$+\frac{\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^j(-\alpha)^p(-kp)^i(-a)^s ab\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^n j!\,p!\,i!\,s!}\int_0^{x_\circ}x^{r+b+j+i+bs-1}e^{anx^b}dx \tag{32}$$

$$e^{anx^b}=\sum_{q=0}^{\infty}\frac{(anx^b)^q}{q!} \tag{33}$$

From substituting Eq (33) in Eq(32) we get

$$E(X^r)$$
$$=\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^j(-\alpha)^p(-kp)^i(an)^q\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^n}\int_0^{x_\circ}x^{j+r+i+bq}\,dx$$
$$+\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^j(-\alpha)^p(-kp)^i(-a)^s(an)^q ab\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^n q!\,j!\,p!\,i!\,s!}\int_0^{x_\circ}x^{r+b+j+i+bs+bq-1}\,dx \tag{34}$$

$$E(X^r)=\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(an)^q\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^n}\left[\frac{x^{j+r+i+bq+1}}{j+r+i+bq+1}\right]_0^{x_\circ}+$$
$$\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(-a)^s(a)^{q+1}n^{\,q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^n q!\,j!\,p!\,i!\,s!}\left[\frac{x^{r+b+j+i+bs+bq}}{r+b+j+i+bs+bq}\right]_0^{x_\circ} \tag{35}$$

$$E(X^r)=\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(an)^q\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^n}\left[\frac{x_\circ^{\,j+r+i+bq+1}}{j+r+i+bq+1}\right]+$$
$$\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(-a)^s(a)^{q+1}n^{\,q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^n q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ^{\,r+b+j+i+bs+bq}}{r+b+j+i+bs+bq}\right] \tag{36}$$

Proposition 1.

The mean (μ) for a random variable $\mathbf{X\sim OTIE-W-E\ D}$ is given by :

$$\mu = E(X) = \frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}\left[\frac{x_\circ{}^{j+i+bq+2}}{j+i+bq+2}\right]+$$

$$\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ{}^{1+b+j+i+bs+bq}}{1+b+j+i+bs+bq}\right] \quad (37)$$

Proof: The mean (μ) for a random variable $X\sim OTIE-W-E\ D$ is obtained by putting r=1 in eq.(36)

Theorem 1.

The r^{th} moment a bout the mean for a random variable $X\sim OTIE-W-E\ D$ is given by:

$$E(X-\mu)^{r} = \sum_{t=0}^{r}\binom{r}{t}(-1)^{r-t}\mu^{r-t}\left[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}\left[\frac{x_\circ{}^{j+t+i+bq+1}}{j+t+i+bq+1}\right]+\right.$$

$$\left.\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ{}^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]\right] \quad (38)$$

Proof : The r^{th} moment a bout the mean for a random variable $X\sim OTIE-W-E\ D$ is given by

$$E(X-\mu)^{r}=\int_{0}^{\infty}(x-\mu)^{r}.\,g(x;\alpha,k,a,b)dx \quad (39)$$

Where μ the mean for a random variable $X\sim OTIE-W-E\ D$ and $g(x;\alpha,k,a,b)$is the PDF

by using the binomial series expansion of $(x-\mu)^{r}$ yields:

$$(x-\mu)^{r} = \sum_{t=0}^{r}\binom{r}{t}(-1)^{r-t}\mu^{r-t}x^{t} \quad (40)$$

By substituting Eq (40) in Eq (39) we get:

$$E(X-\mu)^{r} = \sum_{t=0}^{r}\binom{r}{t}(-1)^{r-t}\mu^{r-t}\int_{0}^{\infty} x^{t}.\,g(x;\alpha,k,a,b)dx \quad (41)$$

$$E(X-\mu)^{r} = \sum_{t=0}^{r}\binom{r}{t}(-1)^{r-t}\mu^{r-t}\mu_{t} \quad (42)$$

Here μ_t represents the t th moment of OTIE-W-ED ,by substituting the equation (36) (by replacing r by t) into equation (42), we get the rth moment about the mean of OGE-W-E distribution as follows:

$$E(X-\mu)^{r}$$

$$= \sum_{t=0}^{r}\binom{r}{t}(-1)^{r-t}\mu^{r-t}\left[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}\,k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}\left[\frac{x_\circ{}^{j+t+i+bq+1}}{j+t+i+bq+1}\right]+\right.$$

$$\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]] \quad (43)$$

Proposition 2: The variance for a random variable $X\sim OTIE-W-E\ D$ is given by

$$E(X-\mu)^2$$
$$=\sum_{t=0}^{2}\binom{2}{t}(-1)^{2-t}\mu^{2-t}[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1}q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}[\frac{x_\circ^{j+t+i+bq+1}}{j+t+i+bq+1}]$$

$$+\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]] \quad (44)$$

Proof : The The variance for a random variable X is defined

$$\mathrm{var}(x) = E(X-\mu)^2 \quad (45)$$

The variance for a random variable $X\sim OTIE-W-E\ D$ is given by setting r=2 in Eq (38)

This complete the proof.

THE COEFFICIENT OF SKEWNESS ,KURTOSIS AND VARIATION

The coefficient of skewedness , kurtosis and variation of the TIED as follows

1) The coefficient of skewedness(CS) of the OTIE-W-ED is given by :

$$CS=\frac{A}{B} \quad (46)$$

A

$$=\sum_{t=0}^{3}\binom{3}{t}(-1)^{3-t}\mu^{3-t}[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1}q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}[\frac{x_\circ^{j+t+i+bq+1}}{j+t+i+bq+1}]$$

$$+\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]]$$

B

$$=[\sum_{t=0}^{2}\binom{2}{t}(-1)^{2-t}\mu^{2-t}[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1}q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}[\frac{x_\circ^{j+t+i+bq+1}}{j+t+i+bq+1}]$$

$$+\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]]]^{\frac{3}{2}}$$

2) The coefficient of kurtosis (CK) of the OTIE-W-ED is given by :

$$CK = \frac{C}{D} \qquad (47)$$

C

$$= \sum_{t=0}^{4} \binom{4}{t} (-1)^{4-t} \mu^{4-t} [\frac{\sum_{q=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (an)^{q}\, k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\, j!\, p!\, i! \binom{p+1}{n} (-1)^{n}} [\frac{x_\circ^{j+t+i+bq+1}}{j+t+i+bq+1}]$$

$$+ \frac{\sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (-a)^{s} (a)^{q+1} n^{q} b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2} \binom{p+2}{n} (-1)^{n} q!\, j!\, p!\, i!\, s!} \left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]]$$

D

$$= [\sum_{t=0}^{2} \binom{2}{t} (-1)^{2-t} \mu^{2-t} [\frac{\sum_{q=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (an)^{q}\, k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\, j!\, p!\, i! \binom{p+1}{n} (-1)^{n}} [\frac{x_\circ^{j+t+i+bq+1}}{j+t+i+bq+1}]$$

$$+ \frac{\sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (-a)^{s} (a)^{q+1} n^{q} b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2} \binom{p+2}{n} (-1)^{n} q!\, j!\, p!\, i!\, s!} \left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]]]^{2}$$

3) The coefficient of variation (CV) of the OTIE-W-ED is given by :

$$CV = \frac{E}{F} \qquad (48)$$

E

$$= [\sum_{t=0}^{2} \binom{2}{t} (-1)^{2-t} \mu^{2-t} [\frac{\sum_{q=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (an)^{q}\, k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\, j!\, p!\, i! \binom{p+1}{n} (-1)^{n}} [\frac{x_\circ^{j+t+i+bq+1}}{j+t+i+bq+1}]$$

$$+ \frac{\sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (-a)^{s} (a)^{q+1} n^{q} b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2} \binom{p+2}{n} (-1)^{n} q!\, j!\, p!\, i!\, s!} \left[\frac{x_\circ^{t+b+j+i+bs+bq}}{t+b+j+i+bs+bq}\right]]^{\frac{1}{2}}$$

$$F = \mu = \frac{\sum_{q=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (an)^{q}\, k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\, j!\, p!\, i! \binom{p+1}{n} (-1)^{n}} [\frac{x_\circ^{j+i+bq+2}}{j+i+bq+2}] +$$

$$\frac{\sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} (-k)^{j+i} (-\alpha)^{p} (p)^{i} (-a)^{s} (a)^{q+1} n^{q} b\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+2} \binom{p+2}{n} (-1)^{n} q!\, j!\, p!\, i!\, s!} \left[\frac{x_\circ^{1+b+j+i+bs+bq}}{1+b+j+i+bs+bq}\right]$$

Proof :

1)we can find The coefficient of skewedness(CS) of the OTIE-W-ED by using the following equation

$$CS = \frac{E(X-\mu)^{3}}{[E(X-\mu)^{2}]^{\frac{3}{2}}} \qquad (49)$$

Let B=$[E(X-\mu)^2]^{\frac{3}{2}}$ (50)

We can find CS by finding A and B as follows:

To find A we set r=3 in Eq.(38)

To find B by substituting the equation (38) (by setting r = 2) into equation (50) .

2) we can find The coefficient of kurtosis (Ck) of the OTIE-W-ED by using the following equation

Ck=$\frac{E(X-\mu)^4}{[E(X-\mu)^2]^2}$ (51)

Let c= $E(X-\mu)^4$ (52)

Let D=$[E(X-\mu)^2]^2$ (53)

We can find CK by finding C and D as follows :

To find C we set r=4 in Eq.(38)

To find D by substituting the equation (38) (by setting r = 2) into equation (53) .

3) we can find The coefficient of variation (CV) of the OTIE-W-ED by using the following equation

CV=$\frac{\sqrt{E(X-\mu)^2}}{\mu}$ (54)

Let E=$\sqrt{E(X-\mu)^2}$ (55)

Let F= μ (56)

We can find CV by finding E and F as follows

To find E by substituting the equation(38) (by setting r=2) in Eq.(55)

To find F by substituting the equation (37) into equation (56) .

THE MOMENT GENERATING FUNCTION

The moment generating function of OTIE-W-E distribution is given by the following theorem:

Theorem 2 : The moment generating function $M_x(t)$ of OTIE-W-E distribution is given by:

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^p(p)^i(an)^q\, k\alpha e^{\frac{\alpha}{x_\circ}}}{\sum_{n=0}^{p+1} q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^n}\left[\frac{x_\circ^{\,j+r+i+bq+1}}{j+r+i+bq+1}\right]\right.$$

$$+\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_{\circ}}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_{\circ}^{r+b+j+i+bs+bq}}{r+b+j+i+bs+bq}\right]] \quad (57)$$

Proof :

$$M_x(t)=E(e^{tx})\int_0^{\infty}e^{tx}g(x;\alpha,k,a,b)dx \quad (58)$$

Using series expansion of e^{tx} yields

$$e^{tx}=\sum_{r=0}^{\infty}\frac{(tx)^r}{r!} \quad (59)$$

Substituting Eq (59) into Eq(58) we get

$$M_x(t)=E(e^{tx})=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r g(x;\alpha,k,a,b)dx \quad (60)$$

Substituting Eq (36) into Eq(60) we get

$$M_x(t)=E(e^{tx})=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\frac{\sum_{q=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(an)^{q}k\alpha e^{\frac{\alpha}{x_{\circ}}}}{\sum_{n=0}^{p+1}q!\,j!\,p!\,i!\binom{p+1}{n}(-1)^{n}}\left[\frac{x_{\circ}^{j+r+i+bq+1}}{j+r+i+bq+1}\right]\right.$$

$$+\frac{\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(-k)^{j+i}(-\alpha)^{p}(p)^{i}(-a)^{s}(a)^{q+1}n^{q}b\alpha e^{\frac{\alpha}{x_{\circ}}}}{\sum_{n=0}^{p+2}\binom{p+2}{n}(-1)^{n}q!\,j!\,p!\,i!\,s!}\left[\frac{x_{\circ}^{r+b+j+i+bs+bq}}{r+b+j+i+bs+bq}\right]] \quad (61)$$

ORDER STATISTICS

Suppose a random sample $x_1, x_2, \ldots\ldots x_n$ of size n for the OTIE-W-ED with $G(x;\alpha$,k,a,b) and $g(x;\alpha$,k,a,b). Let $X_{k1}, X_{k2}, \ldots\ldots X_{kn}$ express the congruous order statistics ; then the PDF of X_{kn} as follows :

$$g_{k,n}(x)=\frac{n!}{(k-1)!\,(n-k)!}g(x;\alpha,k,a,b).\,G(x;\alpha,k,a,b)^{k-1}[1-G(x;\alpha,k,a,b)^{n-k}]$$

By the density function of the r-th order statistics we can determine the maximum , and the minimum

If k=1 , then the minimum value is :

$$g_{1,n}(x)=$$

$$\frac{n!}{(1-1)!(n-1)!}\left[\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^{b}})^{-1}}.\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_{\circ}}}\left(1-e^{-ax^{b}}\right)}\left[k+\frac{abx^{b-1}e^{-ax^{b}}}{1-e^{-ax^{b}}}\right]\right].\left[\frac{e^{\frac{-\alpha\, e^{-kx}}{1-e^{-ax^{b}}}}}{e^{\frac{-\alpha}{x_{\circ}}}}\right]^{1-1}.\left[1-\frac{e^{\frac{-\alpha\, e^{-kx}}{1-e^{-ax^{b}}}}}{e^{\frac{-\alpha}{x_{\circ}}}}\right]^{n-1}$$

$$g_{1,n}(x) = \frac{n!}{(n-1)!}\left[\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}.\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}(1-e^{-ax^b})}\left[k+\frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]\right].\left[1-\frac{e^{\frac{-\alpha\; e^{-kx}}{1-e^{-ax^b}}}}{e^{\frac{-\alpha}{x_\circ}}}\right]^{n-1} \qquad (62)$$

If k=n , then the maximum value is : .

$g_{n,n}(x) =$

$$\frac{n!}{(n-1)!(k-k)!}\left[\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}.\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}\left(1-e^{-ax^b}\right)}\left[k+\frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]\right].\left[\frac{e^{\frac{-\alpha\; e^{-kx}}{1-e^{-ax^b}}}}{e^{\frac{-\alpha}{x_\circ}}}\right]^{n-1}.\left[1-\frac{e^{\frac{-\alpha\; e^{-kx}}{1-e^{-ax^b}}}}{e^{\frac{-\alpha}{x_\circ}}}\right]^{n-n}$$

$$g_{n,n}(x) = \frac{n!}{(n-1)!}\left[\frac{e^{-\alpha\, e^{-kx}(1-e^{-ax^b})^{-1}}.\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}\left(1-e^{-ax^b}\right)}\left[k+\frac{abx^{b-1}e^{-ax^b}}{1-e^{-ax^b}}\right]\right].\left[\frac{e^{\frac{-\alpha\; e^{-kx}}{1-e^{-ax^b}}}}{e^{\frac{-\alpha}{x_\circ}}}\right]^{n-1} \qquad (63)$$

ESTIMATION METHODS

The two considered estimation methods (the maximum likelihood estimation and the moment method) are illustrated to estimate the four parameters of OTIE-W-E distribution .

MAXIMUM LIKELIHOOD ESTIMATION

If $x_1, x_{2\,,\ldots\ldots\ldots}x_n$ denote a random sample from the OTIE-W-E distribution , then the Likelihood function is given by:

$$L = \prod_{i=1}^{n} g(x;\alpha) \qquad (64)$$

$$L = \prod_{i=1}^{n}\left[\frac{e^{-\alpha\, e^{-kx_i}(1-e^{-ax_i{}^b})^{-1}}.\,\alpha\, e^{-kx}}{e^{\frac{-\alpha}{x_\circ}}\left(1-e^{-ax_i{}^b}\right)}\left[k+\frac{abx_i{}^{b-1}e^{-ax_i{}^b}}{1-e^{-ax_i{}^b}}\right]\right] \qquad (65)$$

$$L = \prod_{i=1}^{n}\left[\frac{\alpha\, e^{-kx_i}e^{-\alpha\, e^{-kx_i}(1-e^{-ax_i{}^b})^{-1}}e^{\frac{\alpha}{x_\circ}}}{\left(1-e^{-ax_i{}^b}\right)}\right].\prod_{i=1}^{n}\left[k+\frac{abx_i{}^{b-1}e^{-ax_i{}^b}}{1-e^{-ax_i{}^b}}\right]$$

$$L = \frac{\alpha^n e^{\frac{n\alpha}{x_\circ}}e^{-k\sum_{i=1}^{n} x_i}\,.\,e^{-\alpha\sum_{i=1}^{n}\; e^{-kx_i}(1-e^{-ax_i{}^b})^{-1}}}{\prod_{i=1}^{n}\left(1-e^{-ax_i{}^b}\right)}.\prod_{i=1}^{n}\left[k+\frac{abx_i{}^{b-1}e^{-ax_i{}^b}}{1-e^{-ax_i{}^b}}\right] \qquad (65)$$

By taking log for both sides

$$l = n\ln\alpha + \frac{n\alpha}{x_\circ} - k\sum_{i=1}^{n} x_i - \alpha\sum_{i=1}^{n}\; e^{-kx_i}(1-e^{-ax_i{}^b})^{-1} - \sum_{i=1}^{n}\ln\left(1-e^{-ax_i{}^b}\right)$$

$$+\sum_{i=1}^{n}\ln[k+\frac{abx_i^{b-1}e^{-ax_i^b}}{1-e^{-ax_i^b}}] \qquad (67)$$

by taking the partial derivatives of $\boldsymbol{l}$ with respect to the parameters α,a,b and k and setting the result to zero, we get the following equations:

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}+\frac{n}{x_{\circ}}-\sum_{i=1}^{n} e^{-kx_i}(1-e^{-ax_i^b})^{-1}=0 \qquad (68)$$

$$\frac{\partial l}{\partial k}=-\sum_{i=1}^{n}x_i+\alpha\sum_{i=1}^{n} x_i(1-e^{-ax_i^b})^{-1} e^{-kx_i}+\sum_{i=1}^{n}\frac{1}{\left[k+\frac{abx_i^{b-1}e^{-ax_i^b}}{1-e^{-ax_i^b}}\right]}=0 \qquad (69)$$

$$\frac{\partial l}{\partial a}=\alpha\sum_{i=1}^{n} x_i^b e^{-ax_i^b}e^{-kx_i}(1-e^{-ax_i^b})^{-2}-\sum_{i=1}^{n}\frac{x_i^b e^{-ax_i^b}}{(1-e^{-ax_i^b})}+$$

$$\frac{1}{k+\frac{abx_i^{b-1}e^{-ax_i^b}}{1-e^{-ax_i^b}}}\cdot\frac{\left(1-e^{-ax_i^b}\right)\left[-x_i^b abx_i^{b-1}e^{-ax_i^b}+bx_i^{b-1}e^{-ax_i^b}\right]-abx_i^{b-1}e^{-ax_i^b}x_i^b e^{-ax_i^b}}{(1-e^{-ax_i^b})^2}=0 \qquad (70)$$

$$\frac{\partial l}{\partial b}=\alpha\sum_{i=1}^{n}e^{-kx_i}(1-e^{-ax_i^b})^{-2}e^{-ax_i^b}ax_i^b\ln x_i-\sum_{i=1}^{n}\frac{1}{\left(1-e^{-ax_i^b}\right)}[ax_i^b e^{-ax_i^b}\ln x_i+\frac{1}{\left[k+\frac{abx_i^{b-1}e^{-ax_i^b}}{1-e^{-ax_i^b}}\right]}.$$

$$\frac{\left(1-e^{-ax_i^b}\right)[-abx_i^{b-1}e^{-ax_i^b}ax_i^b\ln x_i+e^{-ax_i^b}(ax_i^{b-1}+abx_i^{b-1}\ln x_i)]-abx_i^{b-1}e^{-ax_i^b}+ax_i^b\ln x_i\, e^{-ax_i^b}}{(1-e^{-ax_i^b})^2}$$

$$=0 \qquad (71)$$

We can obtain the MLEs of the parameters α,k,a,b by solving the equations (68) and (71) numerically for α,k,a,b .

APPLICATIONS

In this section, we provide two applications to real data to demonstrate the importance of the OGE-W-E distribution and we will compare OGE-W-E distribution with the following distributions:

Modified Weibull distribution (MWD) with CDF

$F(x;\alpha,\gamma,\beta)=1-e^{-\alpha x-\beta x^{\gamma}}, x>0$

Flexible Weibull (FW) with CDF

$F(x;\alpha,\gamma,\beta,\theta)=1-e^{-e^{(\beta x^{\gamma}+\theta x^{\alpha})}}, x>0.$

Rayleigh Lomax Distribution(RL) with CDF

$F(x\,;\beta,\theta,\lambda)=1-e^{\frac{\beta}{2}(\frac{\theta}{\theta+x})^{-2\lambda}} \quad x\geq 0 \;\; ,\theta,\beta,\lambda>0$

In order to compare the OGE-W-E distribution with the above distributions, the measures of goodness-of-fit including the Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Consistent Akaike Information Criterion (CAIC), and

Bayesian Information Criterion (BIC) are used where :

$AIC=-2\hat{\ell}+2q$, $BIC=-2\hat{\ell}+q\log(n)$

$CAIC=-2\hat{\ell}+2qn/(n-q-1)$, $HQIC=-2\hat{\ell}+2q\log(\log(n))$ (72)

Where $\hat{l}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, n is the sample size and q is the number of parameters. In general, the distribution, which gives the smallest values from the criteria, shows the better fit to the data.

Data Set

[1.901,2.132,2.203,2.228,2.257,2.350,2.361,2.396,2.397,2.445,2.454,2.474,2.518,2.522,2.525, 2.532,2.575,2.614,2.616,2.618,2.624,2.659,2.675,2.738,2.740,2.856,2.917,2.928,2.937,2.937, 2.977,2.996,3.030,3.125,3.139,3.145,3.220,3.223,3.235,3.243,3.264,3.272,3.294,3.332,3.346, 3.377,3.408,3.435,3.493,3.501,3.537,3.554,3.562,3.628,3.852,3.871,3.886,3.971,4.024,4.027, 4.225,4.395,5.020]; n=63

Table 1. *"Parameters Estimates for the Data Set"*

Models	Parameters Estimates			
OTIE-W-E(α,k,a,b)	$\hat{\alpha}$=3.398	$\hat{k}$=0.9	$\hat{a}$=0.992	$\hat{b}$=0.57
MWD (α,β,γ)	$\hat{\alpha}$=0.88	$\hat{\beta}$=0.99	$\hat{\gamma}$=0.57	
FW (α,γ,β,θ)	$\hat{\alpha}$=0.571	$\hat{\gamma}$=0.881	$\hat{\theta}$=0.907	$\hat{\beta}$=0.991
RLD (λ, β, θ)	$\hat{\lambda}$=0.88	$\hat{\theta}$=0.57	$\hat{\beta}$=0.99	

Table 2 . *The Values of Statistics $\hat{l}$,AIC,HQIC,CAIC and BIC.*

Model	$\hat{\ell}$	**AIC**	**BIC**	**CAIC**	**HQIC**
OTIE-W-ED (α,k,a,b)	-54.339	116.678	115.8753	117.3676	110.7189
MWD (α,β,γ)	-273.847	553.694	553.0920	554.1007	549.2246
FWD (α,γ,β,θ)	-5984.986	11977.972	11977.1694	11978.6617	11972.0129
RLD (λ, β, θ)	-8602.387	17210.774	17210.172	17211.1808	17206.3047

CONCLUSIONS

We introduced new continuous distribution called the odd truncated inverse exponential weibull- exponential distribution for modeling life time data. we define the cdf of the odd truncated inverse exponential weibull-exponential distribution by replacing x of the cdf of (TIED)with the with (F(x))/(S(x)) where F(x)is the cdf of Weibull Distribution and S(x) is the survival function of the Exponential distribution . Our proposed distribution is better than other distributions compared with them in fitting real data sets since they have the smallest values of using criteria like the AIC, CAIC, HQIC and BIC .

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